**Last time:** Functions of several variables, limits/cont., partial deriv., higher-order deriv., the chain rule.

**Topics to be covered:**

<table>
<thead>
<tr>
<th>Section</th>
<th>Problems</th>
</tr>
</thead>
<tbody>
<tr>
<td>12.6 Linear Approximations</td>
<td>12.6: 4, 6, 10, 16</td>
</tr>
<tr>
<td>12.7 Gradients and Directional Derivatives</td>
<td>12.7: 4, 8, 10, 17, 18, 19, 22, 26, 36</td>
</tr>
<tr>
<td>12.8 Implicit Functions</td>
<td>12.8: 2, 5, 6, 11</td>
</tr>
</tbody>
</table>
Question-01

Find the indicated derivatives assuming that the function \( f(x, y) \) has continuous partial derivatives.

(a) \( \frac{\partial}{\partial x} f(y^4, x^3) \)

(b) \( \frac{\partial}{\partial x} f(x^2 f(x, t), f(y, t)) \)

(a) Let \( u(x, y) = y^4 \) and \( v(x, y) = x^3 \). Then we have \( 2 = f(u, v) \).

Using the chain rule,

\[
\frac{\partial}{\partial x} = f'_1 \frac{\partial u}{\partial x} + f'_2 \frac{\partial v}{\partial x} = f'_2 \cdot 3x^2
\]

(b) Look at \( f(\frac{x^2 f(x, t)}{u}, \frac{f(y, t)}{v}) \), \( u = x^2 f(x, t) \), \( v = f(y, t) \)

Then

\[
\frac{\partial}{\partial x} f(\frac{x^2 f(x, t)}{u}, \frac{f(y, t)}{v}) = f'_1 \cdot \left[ 2x f(x, t) + x^2 f'_1(x, t) \right]
\]

\[
= f'_1(u, v) \cdot \left[ 2x f(x, t) + x^2 f'_1(x, t) \right]
\]
Find \( \frac{\partial^2}{\partial y \partial x} f(y^2, xy, -x^2) \) in terms of partial derivatives of \( f \).

Let \( u(x, y) = y^2 \), \( v(x, y) = xy \) and \( w(x, y) = -x^2 \). Then we have

\[
\frac{\partial}{\partial y} \left[ \frac{\partial f}{\partial x} \right] = \frac{\partial}{\partial y} \left[ f_1 \frac{\partial f}{\partial x} + f_2 \frac{\partial v}{\partial x} + f_3 \frac{\partial w}{\partial x} \right]
\]

\[
= \frac{\partial}{\partial y} \left[ y \cdot f_2 \right] - \frac{\partial}{\partial y} \left[ 2x \cdot f_3 \right], \quad \text{where}
\]

\[
= f_2 + y \left[ f_{21} \frac{\partial u}{\partial y} + f_{22} \frac{\partial v}{\partial y} + f_{23} \frac{\partial w}{\partial y} \right]
\]

\[
- 2x \left[ f_{31} \frac{\partial u}{\partial y} + f_{32} \frac{\partial v}{\partial y} + f_{33} \frac{\partial w}{\partial y} \right]
\]

\[
= f_2 + y \left[ 2y f_{21} + x f_{22} \right] - 2x \left[ 2y f_{31} + x f_{32} \right]
\]
Question 03

Let \( f(x, y) = \begin{cases} \frac{x^2y - xy^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases} \)

(a) Find \( f_x(0, 0) \) and \( f_y(0, 0) \).
(b) Show that \( f_{xx}(0, 0) = -1 \) and \( f_{yy}(0, 0) = 1 \).
(c) Does part (b) contradict the fact \( f_{xy}(0, 0) = f_{yx}(0, 0) \)?

\[ f_x(0, 0) = \lim_{h \to 0} \frac{f(0 + h, 0) - f(0, 0)}{h} \]
\[ = \lim_{h \to 0} \frac{0/0^2 - 0}{h} = 0 \]

\[ f_y(0, 0) = \lim_{h \to 0} \frac{f(0, 0 + h) - f(0, 0)}{h} \]
\[ = \lim_{h \to 0} \frac{0/0^2 - 0}{h} = 0 \]

(b) First observe that, for \( (x, y) \neq (0, 0) \) we have

\[ f(x, y) = \frac{y(x^4 + 4x^2y^2 - y^4)}{(x^2 + y^2)^2} \quad \text{and} \quad f_y(x, y) = -\frac{x(y^4 + 4x^2y^2 - x^4)}{(x^2 + y^2)^2} \]

From part (a), we conclude that

\[ f_x(x, y) = \begin{cases} \frac{y(x^4 + 4x^2y^2 - y^4)}{(x^2 + y^2)^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases} \]

\[ f_y(x, y) = \begin{cases} -\frac{x(y^4 + 4x^2y^2 - x^4)}{(x^2 + y^2)^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases} \]
$H(x,y)$

So, want to compute $G_y(0,0)$ and $H_x(0,0)$.

$= f_{xy}(0,0)$

$= f_{yx}(0,0)$

\[
\begin{align*}
(i) \quad f_{xy}(0,0) &= G_y(0,0) = \lim_{h \to 0} \frac{G(0,0+h) - G(0,0)}{h} \\
&= \lim_{h \to 0} \frac{-h^5/h^4 - 0}{h} = -1
\end{align*}
\]

\[
\begin{align*}
(k) \quad f_{yx}(0,0) &= H_x(0,0) = \lim_{h \to 0} \frac{H(0+k,0) - H(0,0)}{k} \\
&= \lim_{k \to 0} \frac{k^5/k^4 - 0}{k} = 1
\end{align*}
\]

as desired.

(c) No contradiction, the hypothesis of the theorem fails

$\implies \ f$ and all partials should be cont. at $(0,0)$

But in fact one can show that $f_{xy}$ and $f_{yx}$ are not cont. at $(0,0)$

Computations $\implies$ (Check $g$)

\[
\begin{align*}
\begin{array}{ll}
& \frac{(x^4 + 12x^2y^2 - 5y^4)(x^2y^2)^2 + y(x^4 + 4x^2y^2 - y^4) + 2(2x^2y^2)}{(x^2 + y^2)^{\frac{v}{4}}} \\
\end{array}
\end{align*}
\]

if $(x,y) \neq (0,0)$

if $(x,y) = (0,0)$
\[
\lim_{(x,y) \to (0,0)} f_{xy}(x,y)
\]

- 1

If \((x,y) = (0,0)\)

Check that \(\lim_{(x,y) \to (0,0)} f_{xy}(x,y) \) d.n.e.

Let enough to check \(\lim_{(x,y) \to (0,0)} f_x(x,y)\) and \(\lim_{(x,y) \to (0,0)} f_y(x,y)\)

on \(P_1: x = 0\)

on \(P_2: y = 0\)

and conclude that they are not the same

(this would show that \(\lim_{(x,y) \to (0,0)} f_{xy}(x,y)\) d.n.e..)
Use a suitable linearization to approximate the number $\sqrt[0.93]{e^{0.02}}$

**Solution** (one-variable case)

Recall the equation:

\[ L(x) = f(a) + f'(a)(x-a) \]

where $x_0 = a$.

Thus,

\[ L(x) = f'(a)(x-a) + f(a) \]

The linearization of $f$ near $(a, b)$ is:

\[ L(xy) = f_x(a, b)(x-a) + f_y(a, b)(y-b) + f(a, b) \]

For the approximation, write $f(x, y) \approx L(x, y)$

Let $f(x, y) = \sqrt{x} e^y$. To approximate $\sqrt[0.93]{e^{0.02}} = f(0.93, 0.02)$

we use the linearization of $f$ at $(1, 0)$

Observe:

\[ f_x = \frac{1}{2\sqrt{x}} e^y, \quad f_y = \sqrt{x} e^y, \quad \text{and} \quad f_x(1, 0) = \frac{1}{2}, \quad f_y(1, 0) = 1, \quad f(1, 0) = 1 \]

So,

\[ f(x, y) \approx L(x, y) = \frac{1}{2} (x-1) + 1 \cdot (y-0) + 1 \]

So,

\[ f(0.93, 0.02) \approx \frac{1}{2} (-0.01) + 0.02 + 1 = 1.015 \]

\[ \therefore \sqrt[0.93]{e^{0.02}} \approx 1.015 \]
Consider the function \( f(x, y, z) = \frac{x}{y} - z \) at \( p_0(3, 1, 1) \).

(a) Compute \( \nabla f(p_0) \).

(b) Is there a unit vector \( u \) such that \( D_u f(p_0) = 5 \)? If yes find one, if no prove that it does not exist.

(c) Is there a unit vector \( u \) such that \( D_u f(p_0) = 3 \)? If yes find one, if no prove that it does not exist.

(d) Let \( S \) be a set of all points \( p(x, y, z) \) where \( f \) increases fastest in the direction of the vector \( A = < 2, 1, 2 > \). Describe the set \( S \).

\[
\nabla f (p_0) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \bigg|_{p_0} = \left\langle \frac{1}{y}, -\frac{x}{y^2}, -1 \right\rangle = \left\langle 1, -3, -1 \right\rangle
\]

(b) Since \( f \) is differentiable at \( p_0 \) (check why), for any unit vector \( \vec{u} \),

\[
D_{\vec{u}} f (p_0) = \vec{u} \cdot \nabla f (p_0)
\]

Also recall that the max rate of change is in the direction of \( \nabla f(p_0) \) with magnitude \( |\nabla f(p_0)| \).

the min is \( -|\nabla f(p_0)| \).

Here, we have

\[
-|\nabla f(p_0)| < D_{\vec{u}} f (p_0) < |\nabla f(p_0)| = \sqrt{11} ( < 5 )
\]

so, \( \exists \) no vector \( \vec{u} \) such that \( D_{\vec{u}} f (p_0) = 5 \)

(c) \( 3 \in [-\sqrt{11}, \sqrt{11}] \) \( \Rightarrow \) It is possible to find \( \vec{u} \) with \( |\vec{u}| = 1 \) s.t \( D_{\vec{u}} f (p_0) = 3 \)

\[
3 = \langle u_1, u_2, u_3 \rangle \cdot \langle 1, -3, -1 \rangle \iff u_1 - 3u_2 - u_3 = 3
\]

\[\text{\quad (with } u_1^2 + u_2^2 + u_3^2 = 1)\]

\( \text{take } \vec{u} = <0, -1, 0> \)

(d) Recall: \( f \) increases fastest in the direction of \( \nabla f(p_0) \).

so, the set \( S \) is given by

\[
S = \left\{ p = (x, y, z) \in \text{Dom}(f) \mid \nabla f (p) \parallel \vec{A} = <2, 1, 2> \implies \nabla f (p) = k \cdot \vec{A} \right\}
\]
\[ \langle -1, -\frac{1}{x}, -1 \rangle = \mathbf{v}(p) = \left\langle \frac{1}{y}, -\frac{x}{y^2}, -1 \right\rangle \]

\[ \langle \\rangle \]

\[ \frac{1}{y} = -1, \quad -\frac{x}{y^2} = -\frac{1}{2} \]

\[ \langle \\rangle \]

\[ y = -1, \quad x = \frac{1}{2}, \quad z = t, \quad t \in \mathbb{R} \]

That means:

\[ S = \left\{ \left( \frac{1}{2}, -1, t \right) : t \in \mathbb{R} \right\} \]

defines a line \( l \) with an eqn:

\[ \langle x, y, z \rangle = \left( \frac{1}{2}, -1, 0 \right) + t \left( 0, 0, 1 \right), \quad t \in \mathbb{R} \]

Just rescale \( \mathbb{R}^3 \) by setting \( h = -\frac{1}{2} \).
Suppose that you are climbing a hill whose shape is given by the equation \( z = 1000 - 0.01x^2 - 0.02y^2 \) and you are standing at a point with coordinates \((60, 100, 764)\).

(a) In which direction should you proceed initially in order to reach the top of the hill faster?
(b) If you climb in that direction, at what angle above the horizontal will you be climbing initially?

\[ \nabla f(x, y) = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) \]

\[ \nabla f(60, 100) = \left( -1, 2 \right) \]

(a) The maximum rate of change can be attained in the direction \( \nabla f(60, 100) \) where

\[ \nabla f(60, 100) = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) \bigg|_{(60, 100)} = \left( -1, 2 \right) \]

(b) Recall: \( \cos \theta = \frac{\nabla f(60, 100) \cdot \vec{v}}{|\nabla f(60, 100)| |\vec{v}|} \) ... check
Question-07

Find the equation of the tangent plane to the level surface of the function \( f(x,y,z) = xy/z^2 \) at the point \( P = (1,2,3) \).

**Solution**

Let \( t = f(x,y,z) = xy/z^2 \). Recall \( t = f(x,y,z) \), for each given \( t = t_0 \),

1. can be viewed as a time-parametr.
2. \( \nabla f(p) \) is normal to the level surface of \( f \) through the point \( P \)

Compute \( \nabla f(p) = \left\langle \frac{y}{z^2}, \frac{x}{z^2}, -\frac{2xy}{z^3} \right\rangle \)

\[ = \left\langle 2, 2, -4 \right\rangle \]

Since \( \nabla f(p) \parallel \mathbf{n} \), just take \( \mathbf{n} = \langle 6, 3, -4 \rangle \).

So, An eqn of the tangent plane \( \overline{P} \) at \( (1,2,3) \) is given by

\[ 6(\overline{x} - 1) + 3(\overline{y} - 2) - 4(\overline{z} - 3) = 0 \]

\[ \iff 6\overline{x} + 3\overline{y} - 4\overline{z} = 0 \]
Check that near the point \((1, 0)\) the equation
\[
\sin xy + y \ln x + e^{3x} - 1 = 0
\]
can be solved for \(y\) as a function of \(x\) and find the value of \(\frac{dy}{dx}\) at the given point.

\[
\text{Soln} \quad \text{let} \quad F(x,y) = \sin xy + y \ln x + e^{3x} - 1 \quad \text{with the eqn} \quad F(y) = 0.
\]

It is enough to check

(i) \(F_x \bigg|_{(1,0)} \neq 0\) \quad (ii) All partial deriv exist and are cont at \((1,0)\).

Observe \(F_x \bigg|_{(1,0)} = \left( x \cos(xy) + \ln(x) + xe^{3x} \right) \bigg|_{(1,0)} = 2 \neq 0 \quad \text{(clear)} \)

So, given imp. eqn. can be solved for \(y\) as a func. of \(x\), say \(y = f(x)\) with \(f(1) = 0\)

To find \(\frac{dy}{dx}\), observe that

\[
F < \begin{array}{c}
\frac{\partial}{\partial x} \\
\frac{\partial}{\partial y}
\end{array} \begin{array}{c}
x \\
y - x
\end{array}
\]

Taking deriv. of both sides of \(F = 0\) wrt \(x\) gives
\[
F_x(x,y) + F_y(x,y) \cdot \frac{dy}{dx} = 0
\]

so,
\[
\frac{dy}{dx} = - \frac{F_x(x,y)}{F_y(x,y)} \bigg|_{(1,0)} = - \frac{y \cos(xy) + \frac{3}{e} + ye^{3x}}{x \cos(xy) + \ln(x) + xe^{3x}} \bigg|_{(1,0)} = 0
\]