Topics to be covered:

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Find and classify the critical points of the function

\[ f(x, y) = 3y^2 - 2y^3 - 3x^2 + 6xy. \]

\[ \nabla f(p) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \]

\[ \text{Recall: To classify critical points } p \text{ of } f \]

we need to check the determinant of the Hessian of \( f \):

\[ H_f(p) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}, \quad \det H_f(p) = f_{xx} f_{yy} - f_{xy}^2 \]

\[ \text{Thm} \]

1) \( \det H_f(p) > 0 \) and \( f_{xx}(p) > 0 \) \( \Rightarrow \) \( f \) has a local min at \( p \)

2) \( \det H_f(p) > 0 \) and \( f_{xx}(p) < 0 \) \( \Rightarrow \) \( f \) has a local max at \( p \)

3) \( \det H_f(p) < 0 \) \( \Rightarrow \) \( p \) is a saddle point

4) \( \det H_f(p) = 0 \) \( \Rightarrow \) no \( \nabla f(p) \).

\[ \begin{align*}
\frac{\partial f}{\partial x} &= -6x + by \\
\frac{\partial f}{\partial y} &= 6y - 6y^2 + 6x \\
f_{xx} &= -6, \quad f_{xy} = 6, \quad f_{yy} = 6 - 12y, \quad f_{yx} = 6
\end{align*} \]

\( \Rightarrow \) \( \det H_f(p) = \begin{vmatrix} -6 & 6 \\ 6 & 6 - 12y \end{vmatrix} \)

\[ \begin{align*}
\nabla f(x_0, y_0) &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\left( \begin{array}{l}
6y - 6x = 0 \\
6y - 6y^2 + 6x = 0
\end{array} \right)
\end{align*} \]

\( \text{Observe that (i) gives } y = x \)

and hence eqn (ii) gives

\( 6y - 6y^2 + 6x = 0 \iff y^2 - 2y = 0 \iff y = 0 \text{ or } y = 2 \)

\( \iff (x=0) \quad (x=2) \)

so, the critical points: \( P_1 = (0, 0), \quad P_2 = (x, x) \)

For the classification, check \( \det H_f(P_i) \) for each \( i \):

\[ \det H_f(0, 0) = \det \begin{bmatrix} -6 & 6 \\ 6 & 6 \end{bmatrix} = -72 < 0 \]

So, \( P_1 = (0, 0) \) is a saddle point.
So, $p_1 = (0,0)$ is a saddle point.

- $\det H_f(2,2) = \det \begin{bmatrix} -6 & 6 \\ 6 & -13 \end{bmatrix} > 0$ and $f_{xx}(2,2) < 0$

So, $f$ has a local max at $(2,2)$. 
Question 02

Find the absolute maximum and minimum values of the function

\[ f(x, y) = 2x^2 - 4x + y^2 - 4y + 1 \]

on the triangular region bounded by \( x = 0, y = 2, y = 2x \).

\[ \nabla f(x, y) = \langle 4x - 4, 2y - 4 \rangle \]

so, \( \nabla f(1, 2) = \langle 0, 0 \rangle \) whereas \( (x, y) = (1, 2) \) not in the interior of \( R \).

\( f_x \) and \( f_y \) are defined everywhere, so no singular points.

Check boundaries: \( \partial R = R_1 \cup R_2 \cup R_3 \).

On \( R_1 : y = 2x, 0 < x < 1 \):

\[ f(x, y) = 6x^2 - 12x + 1 =: g(x) \]

\[ y = 2x \]

To locate them:

- check \( g'(x) = 12x - 12 = 0 \) if \( x = 1 \) (not a critical point)
- end points: \( g(0) = 1 \)
  \[ g(1) = -5 \]

On \( R_2 : x = 0, 0 < y < 2 \):

\[ f(x, y) = y^2 - 4y + 1 =: h(y) \], again cont. on \( [0, 2] \).

so, \( h(y) \) has abs. max/min.
so, \( h(y) \) has abs. max/min.

1. Look at \( h'(y) = 2y - 4 = 0 \) whenever \( y = 2 \)

2. \( h(0) = 1 \), \( h(2) = -3 \)

\[
\begin{align*}
\Rightarrow f(0,0) &= 1 \\
\Rightarrow f(0,2) &= -3
\end{align*}
\]

on \( y = 2, \ 0 \leq x \leq 1 \) : \( f(xy) = 2x^2 - 4x - 3 = k(x) \) on \( [0,1] \) on \( \mathbb{R}_y \)

Clearly, \( k(x) \) is cont on \( [0,1] \), so, \( k(x) \) has abs. max/min. by EVT.

To locate them:

1. Look at \( h'(x) = 4x - 4 = 0 \) if \( x = 1 \)

2. \( h(0) = -3 \) and \( h(1) = -5 \)

\[
\begin{align*}
\Rightarrow f(0,2) &= 3 \\
\Rightarrow f(1,2) &= -5
\end{align*}
\]

To sum up, checking those values gives

\( f \) has an abs. max at \( (0,0) \), \( f(0,0) = 1 \)

\( f \) has an abs. min at \( (1,2) \), \( f(1,2) = -5 \).
Question-03

Find the absolute maximum and minimum values of the function

\[ f(x, y, z) = 2x + 3y + z \]

subject to the unit sphere \( x^2 + y^2 + z^2 = 1 \).

\[ g(x, y, z) = 0 \]

**Solution idea:** To understand global behavior of \( f \), study its Lagrangian function and its local behavior.

Use the method of Lagrange multipliers:

Let \( L(x, y, z, \lambda) = f(x, y, z) + \lambda g(x, y, z) \) where \( g(x, y, z) = x^2 + y^2 + z^2 - 1 \)

Lagrangian function of \( f \)

Here, \( L(x, y, z, \lambda) = 2x + 3y + 2z + \lambda (x^2 + y^2 + z^2 - 1) \), \( \lambda \): Lagrange multiplier.

Look at the points \( P \) for which \( \nabla L \bigg|_P = \langle 0, 0, 0, 0 \rangle \) \( \text{(i)} \)

\[ \begin{align*}
0 &= \frac{\partial L}{\partial x} = 2 + 2\lambda x \\
0 &= \frac{\partial L}{\partial y} = 3 + 2\lambda y \\
0 &= \frac{\partial L}{\partial z} = 1 + 2\lambda z \\
0 &= \frac{\partial L}{\partial \lambda} = x^2 + y^2 + z^2 - 1
\end{align*} \]

\( \Rightarrow \quad x = -\frac{1}{\lambda}, \quad y = -\frac{3}{2\lambda}, \quad z = -\frac{1}{2\lambda} \)

Substitute into the last eqn:

\[ \frac{1}{\lambda^2} + \frac{9}{4\lambda^2} + \frac{1}{4\lambda^2} - 1 = 0 \]

\( \Rightarrow \)

\[ \frac{14 - \lambda^2}{4\lambda^2} = 0 \quad \text{with} \quad \lambda \neq 0 \]
which means \[ n = \pm \sqrt{\frac{3}{2}} \] (for \( n = \pm \sqrt{\frac{3}{2}} \))

so the desired points are

\[ P_1 = (x_1, y_1, z_1) = \left( -\frac{2}{\sqrt{3}}, -\frac{3}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right) \]

\[ P_2 = \left( \frac{2}{\sqrt{3}}, \frac{3}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) \] for \( n = -\sqrt{\frac{3}{2}} \)

observe

\[ f(P_1) = -\frac{14}{\sqrt{3}} \quad \text{"abs min"} \]

\[ f(P_2) = \frac{14}{\sqrt{3}} \quad \text{"abs max"} \]
Using the Lagrange multiplier method, find the point \( Q \) on the plane \( P: x + 2y + 2z = 3 \) that is closest to the origin.

\[ P: x + 2y + 2z = 3 \]

It is enough to understand the extreme values of the func

\[ S(x,y,z) = x^2 + y^2 + z^2 \]

subject to the constraint \( x + 2y + 2z = 3 \)

\[
\begin{bmatrix}
  x + 2y + 2z - 3 = 0 \\
  g(x,y,z)
\end{bmatrix}
\]

Use the method of Lag multipliers:

Define the karyon func. per \( S \) by

\[ L(x,y,z,\lambda) = \frac{x^2 + y^2 + z^2}{S} + \lambda\left(\frac{x + 2y + 2z - 3}{g}\right) \]

Then look at the critical points of \( L \):

\[ \nabla L = \langle 0, 0, 0, 0 \rangle \]

\[ \frac{\partial L}{\partial x} = 2x + 2\lambda = 0 \]

\[ \frac{\partial L}{\partial y} = 2y + 2\lambda = 0 \]

\[ \frac{\partial L}{\partial z} = 2z + 2\lambda = 0 \]

\[ \frac{\partial L}{\partial \lambda} = x + 2y + 2z - 3 = 0 \]

Observe that

\[(x) \text{ implies that} \quad x = -\frac{\lambda}{2}, \quad y = -\lambda, \quad \text{and} \quad z = -\lambda \]

Using the last eqn gives \[ \lambda = -\frac{2}{3} \]

the point we need is \[ Q = \left( \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right) \]
Exercises

1. Find the absolute maximum and minimum values of the function

\[ f(x, y) = x^4 + y^2 \]

on the unit disk \( D = \{(x, y) : x^2 + y^2 \leq 1\} \). (Use the Lagrange multiplier method on \( \partial D \).)

2. Let \( f(x, y) = x^3 y^5 \) and \( g(x, y) = x + y \).

(a) Find the absolute maximum of \( f \) subject to the constraint \( g(x, y) = 8 \) using the Lagrange multiplier method.

(b) By restricting \( f \) onto the line \( x + y = 8 \), verify that the value obtained in the first part is indeed the absolute maximum of \( f \), and also that \( f \) does not have any absolute minimum value on \( x + y = 8 \).