12.6 Linear Approximations & Differentiability

Recall that if \( f: \mathbb{R} \to \mathbb{R} \)

\[
f(x) = f(a) + f'(a)(x-a) = L(x) \quad (x \text{ is near } a)
\]

then it is a good approximation if \( f' \) exists (i.e. \( f \) is differentiable at \( a \)).

**Defn:** Given a function \( z = f(x,y) \), “the linearization of \( f \) at \((a,b)\)”

\[
L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)
\]

**Defn:** \( f(x,y) \) is “differentiable” at \((a,b)\) if

\[
\lim_{(h,k) \to (0,0)} \frac{f(a+h, b+k) + f(a,b) + hf_x(a,b) + kf_y(a,b)}{\sqrt{h^2 + k^2}} = 0
\]

Note that “\( f(x,y) \) is differentiable at \((a,b)\)” means

\( z = f(x,y) \) has a non-vertical tangent plane at \((a,b)\).

**Ex:** Find an approximate value for \( f(x,y) = \sin(\pi x y + \ln y) \) at \((0.01, 1.05)\).

It is convenient to use the linearization at \((0,1)\), since the partials can be easily evaluated:

\[
f(0,1) = \sin(\pi \cdot 0.01 + \ln 1) = \sin 0 = 0
\]

\[
f_x(x,y) = \pi \cos(\pi x y + \ln y) \quad \Rightarrow \quad f_x(0,1) = \pi \cos 0 = \pi
\]

\[
f_y(x,y) = (\pi x + 1) \cos(\pi x y + \ln y) \quad \Rightarrow \quad f_y(0,1) = 1 \cdot \cos 0 = 1
\]

**Linearization of \( f(x,y) \) at \((0,1)\):

\[
L(x,y) = f(0,1) + f_x(0,1)(x-0) + f_y(0,1)(y-1)
\]

\[
L(x,y) = \pi x + (y-1)
\]

\[
f(x,y) \approx L(x,y) \quad \text{for} \ (x,y) \text{ is near} \ (0,1)
\]

\[
f(0.01, 1.05) \approx L(0.01, 1.05) = \pi (0.01) + (1.05-1)
\]
Corollary: If a function \( f \) is differentiable at \((a,b)\), then it is continuous at \((a,b)\).

**Theorem (A Mean Value Theorem)**

If \( f_1(x,y) \) and \( f_2(x,y) \) are continuous in a neighborhood of the point \((a,b)\), and if the absolute values of \( h \) and \( k \) are sufficiently small, then there exist numbers \( \Theta_1 \) and \( \Theta_2 \), each between 0 and 1, such that

\[
f((a\Theta_1 b+ h, b+k) - f(a,b) = h f_1(a\Theta_1 b, b+k) + k f_2(a, b+k)\]

**Proof:**

\[
f((a\Theta_1 b+ h, b+k) - f(a,b) = f((a\Theta_1 b+ h, b+k) - f(a,b)) + (f(a,b+k) - f(a,b))\]

Since \( f(x, y) \) is differentiable for \( y \in (a, b+k) \)

By ordinary MVT, there exists \( \Theta \in (0, 1) \) such that

\[
f((a\Theta_1 b+ h, b+k) - f(a,b) = h f_1(a\Theta_1 b, b+k)\]

Similarly, since \( f(y, z) \) is differentiable for \( y \in (a, b+k) \)

there exists \( \Theta_2 \in (0, 1) \) such that

\[
f((a, b+k) - f(a,b) = k f_2(a, b+k)\]

**Theorem:** If \( f_1 \) and \( f_2 \) are continuous in a neighborhood of \((a,b)\) then \( f(x,y) \) is differentiable at \((a,b)\).

**Proof:** We want to show that

\[
\lim_{{(h,k) \to (0,0)}} \frac{f((a+h, b+k) - f(a,b) - h f_1(a,b) - k f_2(a,b))}{\sqrt{h^2 + k^2}} = 0
\]

\[
= \frac{h}{\sqrt{h^2 + k^2}} \left( f_1(a+h, b+k) + k f_2(a, b+k) - h f_1(a,b) - k f_2(a,b) \right)
\]

By MVT

\[
= \frac{h}{\sqrt{h^2 + k^2}} \left( f_1(a+h, b+k) - f_1(a,b) + k \left( f_2(a, b+k) - f_2(a,b) \right) \right)
\]
\[
\lim_{h \to 0} \frac{f(x+h, y+h) - f(x, y)}{h} + \frac{1}{\sqrt{h^2+k^2}} \left( f_1(x+h, y, z) - f_1(x, y, z) \right) + \frac{1}{\sqrt{h^2+k^2}} \left( f_2(x, y+h, z) - f_2(x, y, z) \right)
\]

As \((h, k) \to (0, 0)\) \(\to 0\)

\(= f_1(x, y, z) \text{ and } f_2(x, y, z) \text{ are continuous around } (x, y, z) \quad \Box\)

**Recall the example**

Note that \(f_1, f_2\) may exist, still \(f\) may not be differentiable (even if it may not be cont.) at \((x, y, z)\)

\[
f(x, y) = \begin{cases} 
\frac{2xy}{x^2+y^2} & (x, y) \neq (0, 0) \\
0 & \text{otherwise}
\end{cases}
\]

We have shown that \(f\) is not cont. at \((0, 0)\) (so it is not differentiable at \((0, 0)\))

But \(f_1(0,0) = 0\) \(\quad \text{exist}\)

\(f_2(0,0) = 0\) \(\quad \text{exist}\)