Lecture 7. Late 17th and 18th century

Jacob Bernoulli (1654–1705) The Art of Conjecture 1713: probability theory, expected values, permutations and combinations, Bernoulli trials and distribution, Bernoulli Numbers and polynomials, proved the Law of Large Numbers. He invented polar coordinates, first used word “integral” for the area under a curve and convinced Leibniz to change the name of the new math from calculus summatorius to calculus integralis. He found the limit (notation “e” was proposed later by Euler). He studied differential equations, like “Bernoulli equations”.

Transcendental curves: the catenary (hanging chain), the brachistochrone (shortest time) led to Calculus of variations.

Nicolaus Bernoulli, brother of Jakob-Johann, 1662–1716 painter and alderman of Basel

Nicolaus I Bernoulli (1687–1759) son of Nicolaus, thesis on probability theory in law. 1716 Galileo chair in Padua, dif.equations and geometry, 1722 Chair in Logic at Basel University, stated in his letters St.Petersburg Paradox, in 1712-14 discussed divergence of (1+x)^n 1742-43 in a letter to Euler criticized his solution of Basel Problem, found error in Newton’s treating of higher derivatives.

The Doctrine of Chances was the second book on probability theory (the first was Cardano’s): discussed the normal distribution; de Moivre was the first to postulate the Central Limit Theorem, and to prove results on the Poisson distribution: de Moivre’s formula (cos x + i sin x)^n = cos(nx) + i sin(nx).

Relates trigonometry and calculus. De Moivre gave Binet’s formula for Fibonacci numbers F_n.

1733 de Moivre gave “Stirling formula” n! = cn^{n+1/2}e^{-n} and approximated constant c, while James Stirling found that c was v(2πn)

Stirling’s Inequality for n!

\[ n! \sim \sqrt{2\pi n} \left( \frac{n}{e} \right)^n \]

Johann Bernoulli (1667–1748) student - Leonhard Euler

Paris Academy’s biennial prize winner in 1727, 1730, and 1734, gave private calculus lessons to Marquis de L’Hôpital in 1691, who published a book in 1696 based on Johann’s lectures containing l’Hôpital’s Rule (that was found by Johann).

1696: Brachistocrone problem stated in Acta Eruditorum by Johann and solved independently by Newton, Leibniz, Jakob Bernoulli, Tschirnhaus and L’Hôpital.

Invented analytical trigonometry. In 1738 plagiarized from his son Daniel by writing a book Hydroaica in which the date was shown 2 years earlier than it was written.

The red brachistochrone (inversed cycloidal) curve is the curve of fastest descent between two points.

Children of Johann

Nicolaus II Bernoulli (1695–1726) curves, dif. equations, probability, fluid dynamics, 1719: Chair in Math in Padua, posed the problem of reciprocal orthogonal trajectories, 1725: St.Petersburg, died after 8 months from appendicitis.

Daniel Bernoulli (1700–1782) friend of Euler, probability (mortality statistics: efficiency of vaccination), economics: risk aversion and risk premium, 1738 fluid mechanics and kinetic theory of gases in Hydrodynamica, vibrating string P + \frac{1}{2} \rho v^2 = constant.

Johann II Bernoulli (1710–1790) propagation of heat and light, magnets, won Paris Academy Prize 4 times, appointed to his father’s chair at Basel

On the Jacob Bernoulli tomb: “Eadem Mutata resurgo” I shall arise the same, though changed in...
Giovanni Girolamo Saccheri (1667-1733) Italian Jesuit priest, scholastic philosopher, and mathematician primarily known for a publication in 1733 related to non-Euclidean geometry (similar to the 11th Century work of Omar Khayyám which was ignored until recently). The Saccheri quadrilateral is now sometimes referred to as the Khayyam-Saccheri quadrilateral.

George Berkeley (1685-1753), Bishop, philosophic theory of "immaterialism" or "subjective idealism", 1734 *The Analyst*, with critics of the foundations of calculus.

Brook Taylor (1685-1731) best known for Taylor's theorem and the Taylor series; in book *Methodus Incrementorum Directa et Inversa (1715)* added to math a new branch "calculus of finite differences", invented integration by parts, and stated Taylor's expansion (which was actually used by James Gregory yet, but remained unrecognized till 1772, when Lagrange proclaimed it the basic principle of dif. Calculus). In the same year of 1715 an essay Linear Perspective.

Colin Maclaurin (1698-1746) a special case of the Tailor series was called Maclaurin series because it was used extensively by Maclaurin (but it was not his discovery); Euler–Maclaurin formula, derived Stirling's formula, integration formulas including Simpson's rule. 1748 *Treatise of Algebra*: linear systems with 2 and 3 unknowns preceded by two years Cramer's publication.

Christian Goldbach (1690-1764) Prussian mathematician who also studied law and worked in Russia since 1725, friend of Euler, in a letter to him in 1742 he stated Goldbach's conjecture. In 1737 one of two heads of Russian Acad. Sci., in 1740 moved from Academy to Ministry of Foreign Affairs, got lands and high noble status.

James Stirling (1692-1770 Edinburgh) Stirling numbers, Stirling permutations, and Stirling's approximation; proved the correctness of Newton's classification of cubics; studied convergence of infinite products, Γ-function

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Thomas Bayes (1701-1761) English statistician, philosopher; Bayes' theorem (edited and published after his death by Richard Price).

\[
p(B|A) = \frac{p(A|B)p(B)}{p(A)}
\]

Gabriel Cramer (1704-1752) from Geneva, 1750 treatise on algebraic curves; proved that a curve of the n-th degree is determined by n(n+3)/2 generic points on it; Cramer's rule for

Cramer's paradox: curves of degree \(d\) are determined by \(d(d+3)/2\) points, but two such curves intersect at \(d^2\) points; was resolved by Euler, who noticed that the corresponding linear systems is overdetermined.

\[
d^2 - \left(\frac{d(d+3)}{2} - 1\right) = \frac{(d-1)(d-2)}{2}
\]
Leonhard Euler (pronounced “Oiler”) (Basel, Switzerland, April 15, 1707 – St. Petersburg, Russia, September 18, 1783) was a Swiss mathematician and physicist, considered to be the preeminent mathematician in history; he is also listed in the Guinness Book of Records as the most prolific, with collected works filling about 90 quarto volumes.

Euler developed important concepts and established mathematical theorems in fields as diverse as calculus, number theory, topology, etc. He introduced the fundamental notion of a mathematical function, and set much of the modern mathematical terminology and notation.

### Childhood

Euler spent most of his childhood around Basel. His father was a pastor and a family friend of the Bernoullis, Johann Bernoulli, who was then regarded as Europe’s foremost mathematician, would eventually have a significant influence on Euler. At the age of 13, Euler matriculated at the University of Basel, and graduated two years later. At the age of 14, he was receiving Saturday afternoon lessons from Johann Bernoulli, who quickly discovered his new pupil’s incredible talent for mathematics.

His father wanted him to become a pastor, but Johann Bernoulli intervened, and convinced Euler’s father to let him study mathematics. At the age of 20, entered the Paris Academy competition, where the problem that year was to find the best way to place the masts on a ship. He won second prize but eventually won the coveted prize 12 times in his career.

### St. Petersburg

Around this time Johann Bernoulli’s two sons, Daniel and Nicolaus were working at the Imperial Russian Academy of Sciences in St. Petersburg. In 1726, Daniel recommended that the post in physiology be filled by Euler and he accepted the offer.

Euler was immediately appointed to a position in the mathematics department. He worked with Daniel Bernoulli with whom he often collaborated. Euler mastered Russian and settled into life in St. Petersburg.

The Academy at St. Petersburg was established by Peter the Great and intended to improve education and science in Russia. As a result, it was very attractive to foreign scholars like Euler, thanks to financial resources and a comprehensive library. Euler swiftly rose through the ranks and was made professor of physics in 1731. In 1733, Euler succeeded Daniel Bernoulli as the head of the mathematics department.

In 1734, Euler got married. The young couple bought a house by the River Neva, and had thirteen children, of whom only five survived childhood.

### Berlin

Frederick the Great of Prussia offered him a post at the Berlin Academy, which he accepted. He left St. Petersburg in 1741 and lived in Berlin for 25 years. Euler, who wrote over 360 articles in Berlin, published the two works which he would be most remembered for: the Introductio in analysin infinitum and the Institutiones calculi differentialis.

In addition, Euler was asked to tutor Frederick’s niece. He wrote over 200 letters to her, which were later compiled into a bestselling volume, all across Europe and America. This work contained Euler’s research on physics and mathematics, as well as a valuable insight into his personality and religious beliefs.

### Return to Russia

At the age of 59, Euler accepted an invitation to return to the St. Petersburg Academy.

Euler had to overcome several tragedies in his second stay. A fire in St. Petersburg cost him his home and almost his life. In 1773, he lost his wife of 49 years, and remarried three years later.

In 1783, he suffered a brain hemorrhage and died. His eulogy was written for the French Academy by Marcq de Condorcet, and an account of his life, with a list of his works, by Nikolaus Fuss, Euler’s grandson-in-law and the secretary of the Imperial Academy of St. Petersburg. The mathematician and philosopher Marcq de Condorcet commented:

"...je commence a calculer et a vivre..." (he ceased to calculate and to live).

### The Bernoullis’

The Bernoulli family is a Swiss family of scientists and mathematicians, with a long history of contributions to science. The most prominent members of the family are Johann I Bernoulli, Jacob I Bernoulli, Johann II Bernoulli, and Daniel Bernoulli, who were all mathematicians and physicists.

### A 1753 portrait by Emanuel Handmann.

This portrait suggests problems of the right eye and that Euler was perhaps suffering from strabismus. The left eye appears healthy, as it was a later cataract that destroyed it. He compensated for it with his mental calculation skills and photographic memory. Euler could recall the content of Virgil from beginning to end without hesitation, and indicate the first and last line of every page in the book he used.

### A 19th century engraving by Bangerman after a portrait by Antonon Maria Loggia.

A portrait of Euler by Benjamin Hay. The portrait shows Euler in his later years, wearing a black coat and a white shirt. The background is a simple, unadorned room.
**Euler's discoveries**

Euler worked in almost all areas of mathematics: geometry, calculus, trigonometry, algebra, and number theory, not to mention continuum physics, lunar theory, etc. His importance in the history of mathematics cannot be overstated: his works correspond to about 80 quarto volumes.

**Mathematical notation**

Euler introduced and popularized several notations that are still in use today, including the concept of a function, the notation for the derivative of a function, and the Greek letter $\pi$ for the ratio of a circle’s circumference to its diameter. He introduced the letter $e$ for the base of the natural logarithm (Euler's number), $\phi$ for the golden ratio, $\theta$ for an angle, and $\Sigma$ for the summation sign. He also introduced the notation $i$ for the imaginary unit $\sqrt{-1}$ and $j$ for the second imaginary unit $\sqrt{-1}$ in electrical engineering.

**Analysis**

Euler introduced the concept of a function $f(x)$ and $g(x)$ to describe how one quantity depends on another. He also introduced the notation $\int f(x) \, dx$ for the indefinite integral of a function $f(x)$ with respect to the variable $x$. He is also known for his work on infinite series, including the famous Euler's identity $e^{ix} = \cos x + i \sin x$.

**Euler's Identity**

Euler's identity is a fundamental equation in mathematics and physics, linking the five most important numbers in mathematics: 0, 1, $e$, $i$, and $\pi$. It states that $e^{i\pi} + 1 = 0$.

**Number theory**

Euler made contributions to the study of numbers, particularly prime numbers. He proved the infinitude of primes, showed that the sum of the reciprocals of the primes diverges, and proved Fermat's Little Theorem. He also introduced the notation $\varphi(n)$ for the number of positive integers less than or equal to $n$ that are coprime to $n$.

**Graph theory, topology, combinatorics, operations research**

Euler developed the concept of a graph, a structure consisting of vertices and edges, to solve the famous Seven Bridges of Königsberg problem. He showed that a graph is Eulerian if and only if it is connected and has at most two vertices with odd degree.

**Applied mathematics, numerical analysis**

Euler developed the concept of a function and its derivatives to solve problems in physics and mechanics. He introduced the notation $f(x)$ for the value of a function $f$ at $x$. He also developed numerical methods for solving differential equations, such as Euler's method.

**Physics and astronomy**

Euler is known for his work in physics and astronomy, including the development of the calculus of variations, which is used to solve problems in physics and engineering. He also made contributions to the theory of elasticity and the investigation of the stability of the solar system.

**Logic**

Euler is credited with the invention of the Venn diagram, a graphical representation of sets and their relationships. The Euler diagram is used in logic to illustrate syllogistic reasoning. These diagrams are now known as Euler diagrams, and are used to show all possible intersections.
Leonhard Euler 1707–1783 one of the greatest mathematicians and the most prolific one in history (his papers were published for 50 years after his death); also physicist, astronomer, logician and engineer. Worked in all the known subjects: infinitesimal calculus, trigonometry, algebra, geometry, number theory, logic. In 1735 solved Basel Problem.

Important books: 1736 Mechanica is the first mechanics textbook based on differential equations. 1748 Introductio in analysin infinitorum (on functions), 1755 Institutiones calculi differentialis (on differential calculus).

Graph theory: 1736 problem known as the Seven Bridges of Königsberg, the formula V − E + F = 2

Math analysis, analytic number theory, theory of hypergeometric series, q-series, hyperbolic trigonometric functions and the analytic theory of continued fractions. He proved the infinitude of primes using the divergence of the harmonic series, and he used analytic methods to gain some understanding of the way prime numbers are distributed the sum of the reciprocals of the primes diverges.

Algebra: a new method for solving quartic equations, four-square identity

\[ \alpha^\phi(n) \equiv 1 \pmod{n} \]

Number theory: he proved that the relationship shown between perfect numbers and Mersenne primes earlier proved by Euclid was one-to-one (known as the Euclid–Euler theorem). He conjectured the law of quadratic reciprocity, proved Newton’s identities, Fermat’s little theorem, Fermat’s theorem on sums of two squares. He invented the Euler function \( \phi(n) \), gave numerous applications of the Bernoulli numbers, Venn diagrams, continued fractions.

Calculus of variations including the Euler–Lagrange equation.

Fourier series: \( \pi/2-x/2=\sin x+3\sin 2x/2+5\sin 3x/3+... \)

Terminology and notation: \( f(x) \) in 1734, \( e \) in 1727, \( \pi \) in 1755, \( i \) in 1777, also \( \Sigma, \Delta y, \Delta^2y \), introduced zeta-function in 1737, gamma-function, Euler function, etc.

Mechanics, fluid dynamics, optics, continuum physics, lunar theory

Music theory 1739 he wrote the Tentamen novae theoriae musicae, hoping to eventually incorporate musical theory as part of mathematics. This part of his work, however, did not receive wide attention and was once described as too mathematical for musicians and too musical for mathematicians

1720-23 Ms. Phil. at the University of Basel, student of Johann Bernoulli, 1726 thesis “De Sono” on the propagation of sound, failed to get a position in Basel “too young”, 1727 took second place in a competition of Paris Academy of Sci. “to find the best way to place the masts on a ship” (later Euler won such annual prize twelve times). 1727 came to St.Petersburg Academy of Sci. to join Daniel Bernoulli after his brother Nicolas died (first Euler took a position in Basel “too young”, failed to get a position in Basel “too young”), returned to Russia by invitation of Catherine the Great (huge annual salary, a pension for his wife, and the promise of high-ranking appointments for his sons). Joke with Diderot: “Sir, \( \alpha+\beta^2/\gamma \) pronunciation, hence God exists—reply!” 1777 found “Cauchy-Riemann equations” (d’Alembert did it in 1752 yet)
1702 David Gregory’s popular account of Newton’s theories.
1707 Newton’s *Arithmetica universalis* (General Arithmetic) with his results in algebra.
1707 De Moivre representation of complex numbers in form \( r \cos x + i \sin x \).
1713 Jacob Bernoulli’s *Ars conjectandi* (The Art of Conjecture) on probability.
1715 Brook Taylor *Direct and Indirect Methods of Incrementation* on calculus.
1718 Jacob Bernoulli’s work on the calculus of variations is published after his death.
1718 De Moivre *The Doctrine of Chances* on probability and statistics.
1719 Brook Taylor publishes *New principles of linear perspective*.
1730 De Moivre gives Stirling’s formula, \( \ln n! = n \ln n - n + O(\ln n) \)
1731 Clairaut publishes *Recherches sur les courbes à double coubure* on spacial curves.
1733 De Moivre describes the normal distribution curve, or law of errors (Gauss did in 1820)
1733 Saccheri: an early work on non-euclidean geometry, hoped to prove the parallel postulate
1734 Berkeley *The analyst*; or a discourse addressed to an infidel mathematician “although the calculus gave true results its foundations were no more secure than those of religion”.
1737 Simpson *Treatise on Fluxions* a textbook with infinite series and integrals of functions.
1739 D’Alembert publishes *Mémoire sur le calcul intégral* (Memoir on Integral Calculus).
1740 Simpson a probability treatise based on the work of de Moivre.
1740 Maclaurin is awarded the Grand Prix for the gravitational theory explaining the tides
1742 Maclaurin *Treatise on Fluxions* the first systematic exposition of Newton’s methods written in reply to Berkeley’s attack on the calculus for its lack of rigorous foundations.
1742 Goldbach conjectures, in a letter to Euler, that every even number \( \geq 4 \) can be written as the sum of two primes. It is not yet known whether Goldbach’s conjecture is true.
1743 D’Alembert *Treatise on Dynamics*: the principle that the internal actions and reactions of rigid bodies in motion are in equilibrium. In 1744 (Treatise on Equilibrium and on Movement of Fluids) he applies his principle to the equilibrium and motion of fluids.
1746 D’Alembert’s theory of complex numbers, attempted to prove the fundamental theorem of algebra
1747 D’Alembert uses partial differential equations to study the winds in Reflection on the General Cause of Winds which receives the prize of the Prussian Academy.
1748 Euler’s *Analysis Infinitiorum* (Analysis of the Infinite) introduced math analysis as the study of functions (rather than on geometric curves, as had been done previously). \( e^{i\pi} = -1 \)
1750 D’Alembert in "three-body problem" applies calculus to celestial mechanics. (Euler, Lagrange and Laplace also work on that problem.)
1750 Cramer a classification of curves, "Cramer’s rule" is given.
1751 Euler theory of logarithms of complex numbers.
1752 D’Alembert: the Cauchy-Riemann equations while investigating hydrodynamics.
1752 Euler theorem \( V = E + F = 2 \) for polyhedra.
1753 Simson in the Fibonacci sequence \( F_n \) the ratio \( F_n/F_{n-1} \) approaches the golden ratio.
1754 Lagrange analysis of autochrone lead to the new subject of the calculus of variations.
1755 Euler In *Institutiones calculi differentialis* introduced the calculus of finite differences. 1757 Lagrange establishes a math society in Italy that become the Turin Academy of Sciences.
1758 "Halley’s comet" on 25 December confirms Halley’s predictions 15 years after his death.
1759 Apéry An Attempt at a Theory of Electricity and Magnetism

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**Jean Baptiste D’Alembert 1717-1783**

Wave equation, fluid mechanics, rival of Clairaut on dynamics, essay on winds, music theory, attempt to prove the Fundamental Theorem of Algebra; 1772 perpetual secretary of Royal Math Soc; 1752 Cauchi-Riemann equations

**D’Alembert’s principle**

D’Alembert’s principle, also known as the Lagrange–D’Alembert principle, is a statement of the fundamental traditional laws of motion. The principle that the resultant of the external forces \( F \) and the kinetic energy acting on a body equals zero. D’Alembert’s principle in mechanics, principle permitting the reduction of a problem in dynamics to one in statics.

**Contribution:**

- Clairaut’s theorem
- Clairaut’s equation
- Clairaut’s relation
- Human computer
Étienne Bézout (1730-1783) 1779 Théorie générale des équations algébriques, theory of elimination and symmetric functions of the roots of an equation; since 1764 used determinants but did not treat the general theory. Bezout theorem (on the intersections), Little Bezout theorem (polynomial remainder theorem). Bezout identity (Lemma): the greatest common divisor is a combination \( ax + by = d \) (proved for polynomials).

Johann Heinrich Lambert (1728-1777) a Swiss polymath credited with the proof that \( \pi \) is irrational (the proof uses generalized continued fraction for \( \tan x \)). Non-Euclidean (hyperbolic) geometry: introduced hyperbolic functions, Lambert quadrilateral, for hyperbolic triangles showed that the sum of angles is less than \( \pi \). Theory of map projections, the first to discuss the properties of conformality and equal area preservation and to point out that they were mutually exclusive; invented seven new map projections. He invented the first practical hygrometer. In 1760, he published a book on photometry, the Photometria.

Alexandre-Théophile Vandermonde (1735-1796) math (also chemist and musician), worked with Bézout, Monge, Lavoisier; on foundations of determinant theory, on the knight tours, combinatorics, symmetric polynomials.

1761 Lambert proves that \( \pi \) is irrational. He publishes it with more general results in 1766.
1763 Monge begins the study of descriptive geometry.
1764 Bayes publishes An Essay Towards Solving a Problem in the Doctrine of Chances which gives Bayes theory of probability. The work contains the important "Bayes' theorem".
1765 Euler: Theory of the Motions of Rigid Bodies the foundation of analytical mechanics.
1766 Lambert: Théorie der Parallellinien, a study of the parallel postulate. In 1767 D'Alembert calls the failure to prove the parallel postulate "scandal of elementary geometry".
1769 Euler publishes the first volume of his three volume work Dioptics.
1769 Euler makes Euler's Conjecture: impossibility that the sum of three fourth powers is a fourth power, the sum of four fifth powers is a fifth power, and similarly for higher powers.
1770 Lagrange proves that any integer can be written as the sum of four squares. In Réflexions sur la résolution algébrique des équations he explains why equations of degrees \( \leq 4 \) can be solved in radicals; his studying of permutations of the roots leads to group theory.
1770 Euler publishes his textbook Algebra.
1771 Lagrange proves Wilson's theorem (first stated without proof by Waring) that \( n \) is prime if and only if \((n - 1)! + 1\) is divisible by \( n \).
1777 Euler used symbol \( i \) for \( \sqrt{-1} \) in a manuscript which will not appear in print until 1794.
1779 Bézout Théorie générale des équation algébriques including "Bézout's theorem".
1780 Lagrange wins the Grand Prix of the Paris Acad. Sci. for his work on perturbations of the orbits of comets by the planets.
1784 Legendre "Legendre polynomials" in his work Recherches sur la figure des planètes on celestial mechanics.
1785 Condorcet publishes Essay on the Application of the Analysis to the Probability of Majority Decisions. It is very important in the study of probability in the social sciences.

Cultural context: Age of Enlightenment (Age of Reason): 18th century, landmarks 1715 (death of Louis 14th) – 1789 (Revolution) Philosophical movement dominated in Europe in the 18th century: ideals of reason as the primary source of authority and legitimacy, also individual liberty, progress, religious tolerance, fraternity, constitutional government and against abuses of the church and state. The most influential publication of the Enlightenment was the Encyclopédie, compiled by Denis Diderot and (until 1759) by Jean le Rond d'Alembert and a team of 150 scientists and philosophers. It was published between 1751 and 1772 in thirty-five volumes, and spread the ideas of the Enlightenment across Europe and beyond.
Joseph-Louis Lagrange, or Giuseppe Lodovico (or Luigi) Lagrangia 1736–1813

analysis, number theory, classical and celestial mechanics, Lagrangian mechanics

(replaced Newtonian)

1754 the problem of tautochrone (the same as brachistocrone), discovered a method of maximizing and minimizing functionals in a way similar to finding extrema of functions, in several letters to Euler in 1754–56 described his results (the calculus of variations, the Euler–Lagrange equations); method of variation of parameters in dif. Equations, the method of Lagrange multipliers

1766 on the proposal of Euler and d’Alembert, Lagrange succeeded Euler as the director of math at the Prussian Acad. of Sci. in Berlin

Works in Celestial mechanics: 1764–66 the libration of the Moon, explained why the same face is turned to the earth; the three-body problem for the Sun, Earth and Moon; the movement of Jupiter’s satellites; 1772–76 stability of Solar system, Lagrangian points,

1788 Analytical mechanics (Mécanique analytique) the most comprehensive treatment of classical mechanics since Newton, formed a basis for math physics in the 19th century

Works in Algebra: 1769–70 a tract on the Theory of Elimination, 1770 every positive integer is the sum of four squares (it was stated by Bachet without justification); 1771 Wilson’s theorem: for any prime number n, n is a prime if and only if n|\(n−1\)! + 1; 1770–71 solving an algebraic equation of any degree via the Lagrange resolvents. 1773 a functional determinant of order 3 (a special case of a Jacobian); the volume of a tetrahedron via the determinant;

1773–77 proved several results enunciated by Fermat; 1775 Recherches d’Arithmetique representations of integers by quadratic forms and by more general algebraic forms: Pell’s equation \(x^2 - ny^2 = 1\) has a nontrivial solution in the integers for any non-square natural number n; Theorie des fonctions analytiques: Lagrange’s theorem the order of a subgroup H of a group G must divide the order of G;

Interpolation and finite differences 1783–93 via Taylor series; the theory of continued fractions, Lagrange polynomials

1787 moved from Berlin to Paris 1794 the first professor of analysis at the École Polytechnique, 1795 Chair at Ecole Normale (after 4 months was closed), 1799 founding member of the Bureau des Longitudes and Senator, under Napoleon: Legion of Honour, 1808 Count of the Empire, buried in the Pantheon.

Joseph Louis Lagrange (1736–1813)

• Born in Turin at that time the capital of Sardinia-Piemont as Giuseppe Lodovico Lagrangia
• 1752 ordained in Turin, moved to Paris, worked in the service of the King as an army officer. He was.py later to become a professor of analysis at the École Polytechnique, 1795 Chair at Ecole Normale (after 4 months was closed), 1799 founding member of the Bureau des Longitudes and Senator, under Napoleon: Legion of Honour, 1808 Count of the Empire, buried in the Pantheon.

Joseph Louis Lagrange (1736–1813)

Method of Lagrange multipliers. To find the maximum and minimum values of \(f(x,y)\) subject to the constraint \(g(x,y) = 0\) (assuming that these extreme values exist and \(\nabla f \neq 0\) on the curve \(g(x,y) = 0\): (a) Find all values of \(x, y\) and \(\lambda\) such that

\[
\nabla f(x,y) = \lambda \nabla g(x,y)
\]

(b) Evaluate \(f\) at all points \((x, y)\) that result from step (a). The largest of these values is the maximum value of \(f\); the smallest is the minimum value of \(f\).

Lagrange’s Mean Value Theorem states that, for any function of a continuous curve, there will always be a point at which the derivative of our curve will be the same as the average curve of the section.

as along algebra and geometry have been separated, their progress have been slow and their uses limited; but when those two sciences have been united, they have lent each mutual forces, and have marched together towards perfection.
Nicolas de Condorcet, Marquis (1743–1794) philosopher, mathematician and political scientist. He wrote the Essay on the Application of Analysis to the Probability of Majority Decisions, Condorcet Paradox, Condorcet method and Condorcet Jury Theorem (concerning voting).

Caspar Wessel (1745-1818, Copenhagen) mathematician and cartographer. In 1799 Wessel gave the geometrical interpretation of complex numbers as points in the complex plane. Since his work was in Danish in a local journal, it went unnoticed for nearly a century; the same results were independently rediscovered by Argand in 1806 and Gauss in 1831.

Jean-Robert Argand (1768-1822) an amateur mathematician, in 1806, while managing a bookstore in Paris, he gave a geometrical interpretation of complex numbers known as the Argand diagram; he is also known for the first rigorous proof of the Fundamental Theorem of Algebra.

Gaspard Monge (1746-1818) developed analytic geometry (point-slope equation) the inventor of descriptive geometry (the mathematical basis of drawing), and the father of differential geometry, because of work *Application de l’analyse à la géométrie* where he introduced the concept of lines of curvature of a surface in 3-dimensional space. During the French Revolution he served as the Minister of the Marine, and was involved in the reform of the French educational system, helping to found the École Polytechnique.

1785 Legendre states the law of quadratic reciprocity but his proof is incorrect.
1785 Lagrange begins work on elliptic functions and elliptic integrals.
1788 Lagrange Analytical Mechanics transforms mechanics into a branch of math. analysis.
1794 Legendre *Éléments de géométrie*, a leading textbook for 100 years that replaced Euclid’s *Elements*.
1796 Laplace nebular hypothesis in *Exposition du systeme du monde* which views the solar system’s origin from the contracting and cooling of a large, flattened, and slowly rotating cloud of incandescent gas.
1796 Gauss gives the first correct proof of the law of quadratic reciprocity
1797 Lagrange *Theory of Analytical Functions* the theory of functions of a real variable, notation $\frac{dy}{dx}$
1797 Wessel vector representation of complex numbers, published in Danish in 1799.
1797 Mascheroni *Geometria del compasso* all Euclidean constructions can be made with compasses alone.
1797 Lazare Carnot treats infinitely small and infinity as limits.
1798 Gauss proves the fundamental theorem of algebra correcting earlier proofs, like d’Alembert’s in 1746.
1799 Monge publishes *Géométrie descriptive*: orthographic projection
1799 Ruffini alg. equations of degree $\geq 5$ cannot be solved in radicals. It was largely ignored as were the further proofs he would publish in 1803, 1808 and 1813.
1801 Gauss *Discourses on Arithmetic* contains seven sections, the first six of which are devoted to number theory and the last to the construction of a regular 17-gon by ruler and compasses.
1801 The minor planet Ceres is discovered but then lost. Gauss computes its orbit from the few observations that had been made leading to Ceres being rediscovered in almost exactly the position predicted by Gauss.
1801 Gauss proves Fermat’s conjecture that every number can be written as the sum of three triangular numbers.
1806 Argand diagram representing complex numbers geometrically in the plane.
1806 Legendre the method of least squares for best approximations to a set of observed data.
1808 Sophie Germain on the *Fermat’s last theorem*, "Germain’s theorem" as it is named by Legendre.
1809 Poincaré discovers two new regular polyhedra.
1809 Gauss used the least-squares method in *Theory of the Movement of Heavenly Bodies*.
Lecture 9. Math in the end of 18th – beginning of 19th centuries

Pierre-Simon, marquis de Laplace (1749-1827) one of the greatest scientists of all time, Newton of France, a count of the First French Empire (1806) and a marquis (1817).

Celestial Mechanics in 5 volumes, (1799–1825) transformed the geometric study of classical mechanics to calculus, developed the nebular hypothesis of the origin of the Solar System and was the first to postulate the existence of black holes and the notion of gravitational collapse.

Statistics: the Bayesian interpretation of probability was developed mainly by Laplace, who developed also the characteristic function as a tool for large-sample theory and proved the first general central limit theorem. De Moivre–Laplace theorem that approximates binomial distribution with a normal distribution.

Calculus, Dif. Equations: Laplace's equation, Laplace transform appear in many branches of mathematical physics, a field that he took a leading role in forming. Laplacian differential operator, General proof of the Lagrange reversion theorem.

Physics: the theory of capillary action and the Young–Laplace equation.

1810-32 Gergonne publishes his mathematics journal Annales de mathématique pures et appliquées which became known as Annales de Gergonne.

1811 Poisson Treatise on Mechanics includes Poisson's work on the applications of math to electricity, magnetism and mechanics.


1814 Argand beautiful proof (with some gaps) of the fundamental theorem of algebra.

1814 Barlow produces Barlow's Tables (factors, squares, cubes, square roots, reciprocals etc. from 1 to 10000).

1815 Peter Roget (the author of Roget's Thesaurus) invents the "log-log" slide rule.

1815 Pfaff publishes important work on what are now called "Pfaffian forms".

1817 Bessel discovers "Bessel functions", in his study of three bodies problem.

1817 Bolzano defines continuous functions without infinitesimals, the Bolzano–Weierstrass theorem.

1818 Adrain publishes Investigation of the figure of the Earth and of the gravity in different latitudes.

1819 Horner "Horner's method" for solving algebraic equations

1820 Brianchon publishes a statement and proof of the nine point circle theorem.
Adrien Marie Legendre (1752-1833)

Number theory: Legendre symbol and partial proof of the quadratic reciprocity law (completed later by Gauss); 1798

Legendre conjecture on existence of primes between \(n^2\) and \((n+1)^2\), statement of the Prime Number Theorem about the distribution of primes (proved later by Hadamard and de la Vallée-Poussin in 1896).

\~1811 introduced the symbol \(\Gamma\) and name “gamma function” such that \(\Gamma(n+1) = n!\).

1830 proved the Fermat’s last theorem for exponent \(n = 5\) (in 1828 also by Lejeune Dirichlet).

Algebra: work on roots of polynomials that inspired Galois theory.

Statistics, approximation: 1806 (appendix to a book on the paths of comets) the least squares method later developed by Gauss (used also in linear regression, signal processing, statistics, and curve fitting).

Elliptic integrals: classified (later completed by Abel). An elliptic integral is an integral of the form

\[
\int \frac{P_0(x) dx}{\sqrt{Q_0(x)}}.
\]

Legendre transformation relates the Lagrangian to the Hamiltonian forms of mechanics, in thermodynamics it is also used to obtain the enthalpy and the Helmholtz and Gibbs energies from the internal energy. Legendre polynomials are solutions to Legendre’s differential equation, which occur frequently in physics and engineering applications, e.g. electrostatics.

1794 Éléments de Géométrie: greatly rearranged and simplified Euclid’s Elements, it became the leading elementary textbook for ~100 years. 

\[
(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)
\]

Paolo Ruffini (1765-1822) an Italian mathematician and philosopher. In 1799 gave an incomplete proof (Abel–Ruffini theorem) that quintic (and higher-order) equations cannot be solved by radicals, and Ruffini’s rule which is a quick method for polynomial division. Ruffini also made contributions to group theory in addition to probability and quadrature of the circle. Ruffini was the first to claim the unsolvability in radicals of algebraic equations higher than quartics: this work was sent to Cauchy in 1799 which was not acknowledged. In 1801-02 it was sent 3 times to Lagrange, in 1803, 1808, 1813 new proofs were published, but nobody wanted to read. In 1821 Cauchy acknowledged correctness.
Jean-Baptiste Joseph Fourier (1768-1830) known for the Fourier series and their applications to problems of heat transfer and vibrations; Fourier transform and Fourier’s law; he is credited for discovery of the greenhouse effect.

1822 The Analytic Theory of Heat developed Newton’s law of cooling, gave heat flow law (Fourier Law)

1831 (edited by Claude-Louis Navier) Fourier’s theorem on the position of the roots of an algebraic equation; the concept of dimensional homogeneity in equations (dimensional analysis); PDE for conductive diffusion of heat.

Siméon Denis Poisson (1781-1840) geometer; a new branch of mathematical physics: the theory of electricity and magnetism.

1813 in Bulletin de la société philomatique Poisson’s equation

Poisson’s equation in probability theory $P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!}$

1831 Poisson derived the Navier–Stokes equations independently of Claude-Louis Navier

Johann Friedrich Pfaff (1765-1825) a precursor of the German school of mathematics, teacher of Carl Friedrich Gauss.
Johann Carl Friedrich Gauss (1777–1855) *Princeps mathematicorum* number theory, algebra, statistics, analysis, differential geometry, geodesy, geophysics, mechanics, electrostatics, astronomy, matrix theory, and optics.

1796 a regular polygon can be constructed by compass and straightedge if and only if the number of sides is the product of distinct Fermat primes and a power of 2. Any number is a sum of 3 triangular numbers: recorded in his diary "ΕΥΡΗΚΑ! num = Δ + Δ + Δ".

1798 *Disquisitiones Arithmeticae* (published in 1801) consolidating number theory as a discipline and has shaped the field to the present day; the quadratic reciprocity law.

1801 a position for Ceres (that was “lost” by astronomers) is predicted very accurately; 1807 was appointed Professor of Astronomy and Director of the astronomical observatory in Göttingen, a post he held for the remainder of his life.

1809 a theory of the motion of planetoids disturbed by large planets, published in *Theory of motion of the celestial bodies moving in conic sections around the Sun* that still remains a cornerstone of astronomical computation; Gaussian gravitational constant, the method of least squares, proved under the assumption of normally distributed errors, Gauss–Markov theorem. The method had been described earlier by Adrien-Marie Legendre in 1805, but Gauss claimed that he had been using it since 1795.

1818 geodesic survey of the Kingdom of Hanover, invented the heliotrope, an instrument that uses a mirror to reflect sunlight over great distances, to measure positions. differential geometry and topology, fields of mathematics dealing with curves and surfaces. Among other things he came up with the notion of Gaussian curvature. 1828 *Theorema Egregium* (remarkable theorem): the curvature of a surface is an intrinsic property (independent of the embedding of a surface into 3-space).

Bolyai’s son, János Bolyai, discovered non-Euclidean geometry in 1829; published in 1832. After seeing it, Gauss wrote to Farkas Bolyai: "To praise it would amount to praising myself. For the entire content of the work ... coincides almost exactly with my own meditations which have occupied my mind for the past thirty or thirty-five years." Gauss was aware of non-Euclidean geometry before it was published by Bolyai, but that he refused to publish any of it because of his fear of controversy.

1831 collaboration with the physicist Wilhelm Weber, study of magnetism (including finding a representation for the unit of magnetism in terms of mass, charge, and time); discovery of Kirchhoff’s circuit laws in electricity. They constructed the first electromechanical telegraph in 1833, which connected the observatory with the institute for physics in Göttingen. Gauss ordered a magnetic observatory to be built in the garden of the observatory, and with Weber founded the "Magnetischer Verein" (magnetic club in German), which supported measurements of Earth’s magnetic field in many regions of the world. He developed a method of measuring the horizontal intensity of the magnetic field which was in use well into the second half of the 20th century, and worked out the mathematical theory for separating the inner and outer (magnetospheric) sources of Earth’s magnetic field.

1840 *Dioptrische Untersuchungen*, images under a paraxial approximation (Gaussian optics): an optical system can be characterized by its cardinal points; Gaussian lens formula.

Gauss was known to dislike teaching and attended only a single scientific conference, which was in Berlin in 1828. However, several of his students became influential mathematicians, among them Richard Dedekind, Bernhard Riemann, and Friedrich Bessel.
The History of Johann Friedrich Carl Gauss
1777 - 1855

Graphical Presentation of Gaussian normal distribution in probability

Found Bell Curve.

Died, February 23, Göttingen, Germany
1855

Life stands before me like an eternal spring with new and brilliant clothes.

Mathematicians stand on each other's shoulders.

Born April 30th, 1777
Brunswick, Germany

Gauß & Wilhelm Weber Invented the first Electric Telegraph.

Contribution

Geometry, Statistics, Number Theory, Planetary Astronomy

The theory of functions, potential theory, optics and geophysics

Findings

1800
Create Method of Least Squares
Mapping the state of Hannover, indispensable tool for analyzing Data

1801
Disquisitiones Arithmeticae published

1801
Found a way to construct the regular with Seventeen Sides

1799
Create Basic Algebra Theorem

1797
Pioneer in Non-Euclidian Geometry

Resources
http://www.famousscientists.org/johann-friedrich-gauss/
http://www.answers.com/Q/Contributions_to_mathematics_by_Carl_Gauss
http://mathworld.wolfram.com/CarlGauss.html
http://www.history.mcs.st-andrews.ac.uk/Quotations/Gauss.html
Baron **Augustin-Louis Cauchy** (1789-1857) a pioneer of analysis, stated and proved theorems of calculus rigorously, rejecting the heuristic principles; founded complex analysis studied of permutation groups in algebra.

"More concepts and theorems have been named for Cauchy than for any other mathematician (in elasticity there are sixteen concepts and theorems named for Cauchy)." Cauchy was a prolific writer: ~800 research articles and 5 complete textbooks.

**Early works:** 1805 solved Problem of Apollonius (find a circle touching three given circles); 1811 generalized Euler’s formula from polyhedra to planar graphs; 1813: the Fermat polygonal number theorem.

**Calculus transformed into Analysis:** 1821 Cours d’Analyse "the man who taught rigorous analysis to all of Europe." Definition of continuity: The function \( f(x) \) is continuous with respect to \( x \) between the given limits if, between these limits, an infinitely small increment in the variable always produces an infinitely small increment in the function itself. Infinitesimals were defined in terms of a sequence tending to zero. The notion of convergence, the theory of series, the theory of functions, differential equations.

**Abstract Algebra:** the theory of permutation groups and substitutions, cycle decomposition theorem, determinants; Cauchy-Binet formula, Cauchy-Schwarz inequality.

**Math Physics:** 1816 memoir on wave propagation (Grand Prix of the French Acad of Sci), theory of light (Fresnel’s wave theory), dispersion and polarization of light; mechanics, the principle of the continuity of matter; equilibrium of rods and elastic membranes, waves in elastic media; in elasticity the theory of stress (3 x 3 symmetric matrix known as the Cauchy stress tensor)

\[
f(a) = \frac{1}{2\pi i} \oint_{C} \frac{f(z)}{z-a} \, dz,
\]

**Complex function theory:** 1814-1825 defined complex numbers as pairs of real numbers Cauchy’s integral theorem. 1826 a residue of a function, poles, 1831 Cauchy’s integral formula, residue theorem. Only in the 1840s the theory started to get response, with Pierre-Alphonse Laurent (1843 Laurent series).

**Augustin Cauchy**

* August 21, 1789 – May 23, 1857
* 1810 - Graduated in civil engineering and went to work as a junior engineer where Napoleon planned to build a naval base
* 1812 – (age 23) Lost interest in engineering, being more attracted to abstract mathematics
* Cauchy had many major accomplishments in both mathematics and science in areas such as complex functions, group theory, astronomy, hydrodynamics, and optics
* Cauchy made 189 contributions to scientific journals
* One of his most significant accomplishments involved determining when an infinite series will converge on a solution
* In wave theory, he defined an empirical relationship between the refractive index and wavelength of light for transparent materials -- Cauchy's Dispersion Equation.
Niels Henrik Abel (1802–1829) proved impossibility of solving the general quintic equation in radicals; worked on the elliptic integrals and elliptic functions, discovered Abelian functions. Abel was unrecognized during his lifetime, lived in poverty and died at age 26.

1824 Memoir on algebraic equations, in which the impossibility of solving the general equation of the fifth degree is proven; only six pages (to save money on printing). 1826 more detailed proof in the first volume of Crelle’s Journal.

1825 received a grant for research travel to Göttingen to visit Gauss and Paris, but first he went to Berlin (4 months) and met August Leopold Crelle, who was about to publish Journal für die reine und angewandte Mathematik and Abel contributed 7 papers to it in its first year. In Leipzig and Freiberg Abel did research in the theory of functions: elliptic (double periodicity), hyperelliptic, and a new class of abelian functions. From Freiberg he went to Dresden, Prague, Vienna, Trieste, Venice, Verona, Bolzano, Innsbruck, Luzern and Basel and finally Paris. 1826 submitted a memoir on addition of algebraic differentials to French Acad. of Sci. It was reviewed by Cauchy, but was put aside and forgotten until his death. Notes: “Paris mathematicians are interested only in astronomy, heat theory, optic and elasticity. Cauchy is the only pure mathematician, although he is absolutely insane and there is nothing to do with it.” 1827 return to Berlin, declined to work as an editor of Crelle’s Journal and was back in Norway. His tour was viewed as a failure: not visited Gauss in Göttingen, not published anything in Paris, so, his scholarship was not renewed and he started tutoring while sending most of his work to Crelle’s Journal. While in Paris, Abel contracted tuberculosis and died just two days before a letter from Crelle, informing that he found him a position of Prof. at the Univ. of Berlin.

Évariste Galois (1811–1832) found necessary and sufficient condition for a polynomial to be solvable by radicals. His work laid the foundations for Galois theory and group theory, two major branches of abstract algebra. He died at age 20 from wounds suffered in a duel.

1829 submitted two papers on the polynomial equations to the Academy of Sciences, but Cauchy who refereed these papers, did not accept them for publication, although recognized the their importance and suggested to combine the two papers into one and to submit it for the Academy’s Grand Prize. In 1830, Galois submitted his work to the Academy’s secretary Fourier, for the Grand Prix, but Fourier died soon and the manuscript was lost. Galois published three other papers that year, one of which laid the foundations for Galois theory, the second one was about finding the roots of equations, and from the third a notion of finite field was developed.

Galois was the first to use the word group in a modern sense, he developed the concept of normal subgroup. Finite fields are called now Galois field since he introduced them and understood like in modern times.

In his last letter to Chevalier and attached manuscripts, the second of three, he made basic studies of linear groups over finite fields: constructed the linear group $GL(v, p)$ over a prime field, as well as the projective group $PSL(2, p)$ viewed via fractional linear transforms, and observed that they were simple except if $p$ was 2 or 3.

1827 developed interest for mathematics
1828 failed the entrance exam to the Ecole Polytechnique, but continues to work on his own
1829 first mathematics paper published on continued fractions in the Annales de mathématiques
25 May and 1 June he submitted articles on the algebraic solution of equations to the Académie des Sciences. Cauchy was appointed as referee of Galois paper.
1829 entered the Ecole Normale.
1830 learned of a posthumous article by Abel which overlapped with a part of his work and submitted a new article On the condition that an equation be soluble by radicals in February. The paper was sent to Fourier, the secretary of the Academy, to be considered for the Grand Prize in mathematics. Fourier died in April 1830 and Galois paper was never subsequently found and no longer considered for the prize which went to Abel and Jacobi.
1830 he published three papers in Bulletin de l'Académie
1830 Galois was invited by Poisson to submit a third version of his memoir on an equation
1830 For writing a political letter Galois was expelled and he joined On 31 December 1830 the Artillery of the National Guard which was subsequently was abolished by Royal Decree since the new King Louis-Philippe let it as a threat to the throne.
1831 In and out of prison
1832 Galois contracted cholera during the Paris epidemic. He apparently fell in love with Stephen-Palais du Motel, the daughter of his physician.
1832 Galois fought a duel with Percheaux d'Hertingville over the priest's physician's daughter named Stephanie-Felice du Motel; abandoned by both Percheaux as well as his seconds. A peasant took him to a hospital, where he died at the age of 21 in 1832.