M E T U
Department of Mathematics

Group	CALCULUS II Mid Term 1	List No.

Code : *Math 120* Acad. Year : *2005-2006* Semester : *Spring* Coordinator : *Muhiddin Uguz*	Last Name : Name : Student No. : Department : Section : Signature :
Date : *April.1st.2006* Time : *09:30* Duration : *120 minutes*	6 QUESTIONS ON 5 PAGES TOTAL 60 POINTS

1	2	3	4	5	6		

Question 1 (4+4+4=12 points) For the series $\sum_{n=1}^{\infty} \frac{(3x+2)^n}{\sqrt{n}}$ $= \sum_{n=1}^{\infty} \frac{(3(x+\frac{2}{3}))^n}{\sqrt{n}}$

a)Find the radius R of convergence

$$= \sum_{n=1}^{\infty} \frac{3^n}{\sqrt{n}} (x - (-\tfrac{2}{3}))^n \text{ is a power serie around } -2/3$$

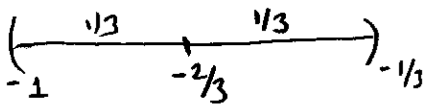
$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{3^{n+1}}{\sqrt{n+1}} |x+\tfrac{2}{3}|^{n+1} \cdot \frac{\sqrt{n}}{3^n} \frac{1}{|x+\frac{2}{3}|^n}$$

$$= 3|x+\tfrac{2}{3}| \lim_{n\to\infty} \sqrt{\frac{n}{n+1}} = 3|x+\tfrac{2}{3}|$$

Given power serie is convergent if $|x+\frac{2}{3}| < 1/3$

so $R = 1/3$

b)Find the interval I of convergence



$x=-1 \Rightarrow \sum_{n=1}^{\infty} \frac{(3x+2)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ alternating series, $a_n = \frac{1}{\sqrt{n}}$ decreasing, $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ } convergent series (by Alternating Series Test)

$x = \frac{-1}{3} \Rightarrow \sum_{n=1}^{\infty} \frac{(3x+2)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ div. (by p-test) $\qquad I = [-1, -1/3)$

c)If $f(x) = \sum_{n=1}^{\infty} \frac{(3x+2)^n}{\sqrt{n}}$, $x \in I$, find 2006$^{\text{th}}$derivative $f^{(2006)}(\frac{-2}{3})$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(-2/3)}{n!} (x+\tfrac{2}{3})^n = \overset{0}{\overbrace{f(-\tfrac{2}{3})}} + \sum_{n=1}^{\infty} \frac{f^{(n)}(-2/3)}{n!} (x+\tfrac{2}{3})^n$$

$$= \sum_{n=1}^{\infty} \frac{3^n\, n!}{\sqrt{n}\, n!} (x+\tfrac{2}{3})^n$$

$$\Rightarrow f^{(2006)}(-2/3) = \frac{3^{2006} (2006)!}{\sqrt{2006}}$$

Question 2 (4+4+4=12 points)

For each series below determine whether it is: absolutely convergent, conditionally convergent, or divergent. You must **verify** your answers, **state** any theorem you use.

a) $\sum_{n=1}^{\infty} e^{a_n}$, where $\sum_{n=120}^{\infty} a_n$ is convergent

Since $\sum a_n$ is conv., by n^{th} term test. $\lim_{n\to\infty} a_n = 0$

$$\Rightarrow \lim_{n\to\infty} e^{a_n} \underset{\uparrow}{=} e^{\lim_{n\to\infty} a_n} = e^0 = 1 \neq 0 \Rightarrow \text{by } n^{th} \text{ T.T, } \sum e^{a_n} \text{ is divergent}$$

e^x is continuous

b) $\sum_{n=1}^{\infty} \frac{\cos(n)+e^{-n}}{n^3-2n+4}$

$$0 \le \left|\frac{\cos(n)+e^{-n}}{n^3-2n+4}\right| \le \frac{1}{|n^3-2n+4|}\left(\overset{\le 1}{|\cos(n)|} + \overset{\le 1}{|e^{-n}|}\right) \le \frac{2}{|n^3-2n+4|}$$

$$\underset{\uparrow}{=} \frac{2}{n^3-2n+4} = b_n$$

for large n's.

Since $\lim_{n\to\infty} \frac{b_n}{\frac{1}{n^3}} = 2$, we have $\sum b_n$ is conv by limit Comparison test (as $\sum \frac{1}{n^3}$ is convergent)

Therefore $\sum \frac{\cos(n)+e^{-n}}{n^3-2n+4}$ is absolutely convergent (by Comparison Test)

c) $\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$

let $f(x) = \frac{1}{x\ln(x)}$ then

1) $f(x)$ is continuous on $[2,\infty)$

2) $f'(x) = -(\ln(x)+1)/(x\ln(x))^2 < 0$ on $[2,\infty)$

hence $f(x) \searrow$ on $[2,\infty)$

3) $\lim_{x\to\infty} \frac{1}{x\ln(x)} = 0$.

Thus by integral Test, $\sum_{n=2}^{\infty} \frac{1}{n\ln n}$ is convergent if and only if $\int_2^{\infty} \frac{1}{x\ln x}\,dx$ is convergent

$$\int_2^{\infty} \frac{1}{x\ln x}\,dx = \lim_{c\to\infty} \int_2^c \frac{1}{x\ln x}\,dx = \lim_{c\to\infty} \int \frac{1}{u}\,du = \lim_{c\to\infty} \ln|u|$$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$$= \lim_{c\to\infty} \ln(\ln x)\Big|_2^c = \lim_{c\to\infty} \ln(\ln c) - \ln(\ln 2)$$

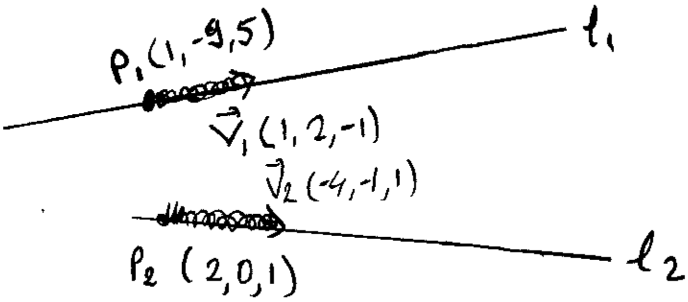
$\to \infty$ divergent,

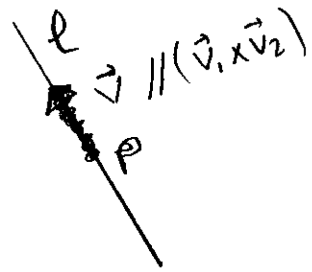
Hence $\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$ is divergent

Question 3 (6+6=12 points)

Let $L_1 = \begin{cases} x = 1+t \\ y = -9+2t \quad t \in R \\ z = 5-t \end{cases}$ and $L_2 = \begin{cases} x = 2-4s \\ y = -s \quad s \in R \\ z = 1+s \end{cases}$

a) Find an equation of the line L which is perpendicular to both L_1 and L_2 and passes through the point of intersection of L_1 and L_2.



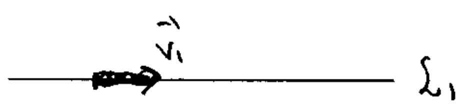


$\underline{\underline{\vec{v}}} = \vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ -4 & -1 & 1 \end{vmatrix} = i(2-1) - j(1-4) + k(-1+8) = (1, 3, 7)$

$\underline{\underline{P}} = L_1 \cap L_2 \qquad \left.\begin{matrix} 1+t = 2-4s \\ -9+2t = -s \end{matrix}\right\} \left.\begin{matrix} s=-1 \\ t=5 \end{matrix}\right\} \Rightarrow P = (6, 1, 0)$

$\underline{\underline{L}}: \quad \begin{matrix} x = 6 + 1t \\ y = 1 + 3t \\ z = 0 + 7t \end{matrix} \qquad t \in \mathbb{R}$

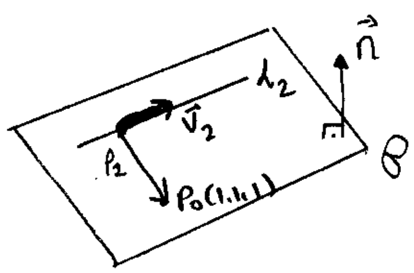
b) Find the distance between the line L_1 and the plane containing the point $P_0(1,1,1)$ and the line L_2



$\vec{n} \parallel (\vec{v}_2 \times \vec{P_2P_0}) = \begin{vmatrix} i & j & k \\ -4 & -1 & 1 \\ -1 & 1 & 0 \end{vmatrix}$

$= i(-1) - j(1) + k(-5)$

$= (-1, -1, -5)$



Note that

$\vec{n} \cdot \vec{v}_1 = (-1,-1,-5) \cdot (1,2,-1) = -1-2+5 = 2 \neq 0$

$\vec{P_2P_0} = (-1, 1, 0)$

so $\vec{n} \not\perp \vec{v}_1 \Rightarrow L_1 \not\parallel P \Rightarrow L_1 \cap P \neq \emptyset$

$\Rightarrow$ distance is zero

(or/ simply, as part (a) indicates, L_1 & L_2 intersects at $P(6,1,0)$. Hence distance is zero)

Question 4 (4+4=8 points) Determine whether the following limits exist:

a) $\lim_{(x,y)\to(1,2)} \frac{(x-1)(y-2)}{(x-1)^2+(y-2)^2}$ $= \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$

along $y=mx$, $\lim_{x\to 0} \frac{mx^2}{x^2+m^2x^2} = \frac{m}{1+m^2}$ depends on m

so limit does not exist

b) $\lim_{(x,y)\to(0,0)} \frac{xy^5}{x^2+y^2}$

$$0 \le \left|\frac{xy^5}{x^2+y^2}\right| = |xy^3| \frac{y^2}{x^2+y^2} \le \underbrace{|xy^3|}$$

as $(x,y)\to(0,0)$

0

$\Rightarrow$ by squeezing rule,

$\lim_{(x,y)\to(0,0)} |f(x,y)| = 0$

hence

$\lim_{(x,y)\to(0,0)} f(x,y) = 0$

Question 5 (6 points) Find an approximation of $\int_0^2 \frac{\sin x}{x} dx$ with an error less than $\frac{1}{1000}$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad \forall x \in \mathbb{R} \Rightarrow \frac{\sin x}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} \quad \forall x \in \mathbb{R}$$

$$\Rightarrow \int_0^2 \frac{\sin x}{x} dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!} \Bigg|_0^2 = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{(2n+1)(2n+1)!}$$

$$\approx 2 - \frac{2^3}{3.3!} + \frac{2^5}{5.5!} - \frac{2^7}{7.7!}$$

$$|\text{error}| < \frac{2^9}{9.9!} < 10^{-3}$$

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Question 6 (5+5=10 points) Let $f(x,y) = \begin{cases} \frac{x^2-2y^2}{x-y} & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$

a) Compute $f_x(0,0) = \frac{\partial f}{\partial x}(0,0)$

$$\lim_{t \to 0} \frac{f(t,0) - \overbrace{f(0,0)}^{=0}}{t} = \lim_{t \to 0} \frac{\frac{t^2}{t}}{t} = \lim_{t \to 0} 1 = 1$$

Thus $f_x(0,0) = 1$

b) Is $f_x(x,y)$ continuous at $(0,0)$?

if $x \neq y$ then $f_x(x,y) = \frac{2x(x-y) - (x^2-2y^2)}{(x-y)^2}$

Thus $= \frac{x^2+2y^2-2xy}{(x-y)^2}$

$$\lim_{\substack{(x,y)\to(0,0) \\ \text{along } y = mx \\ \text{with } m \neq 1}} f(x,y) = \lim_{x \to 0} \frac{x^2 + 2m^2x^2 - 2x\,mx}{(x-mx)^2}$$

$$= \lim_{x \to 0} \frac{\cancel{x^2}(1 + 2m^2 - 2m)}{\cancel{x^2}(1-m)^2} = \frac{1+2m^2-2m}{(1-m)^2}$$

depends on m

so limit does <u>not</u> exist.

Hence $f_x(x,y)$ is <u>not</u> continuous at $(0,0)$

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Group	CALCULUS II Mid Term 2	List No.

Code : *Math 120* Acad. Year : *2005-2006* Semester : *Spring* Coordinator: *Muhiddin Uguz*	Last Name : Name : Student No. : Department : Section : Signature :
Date : *April.29th.2006* Time : *09:30* Duration : *120 minutes*	6 QUESTIONS ON 6 PAGES TOTAL 60 POINTS

1	2	3	4	5	6	SHOW YOUR WORK	

Question 1 (8 points)

Find the set of all points on the ellipsoid $x^2 + 2y^2 + 4z^2 - 12 = 0$ at which the tangent plane to the ellipsoid is parallel to the plane $y + z = 120$.

Let $F(x,y,z) = x^2 + 2y^2 + 4z^2 - 12$. Then $\nabla F(x,y,z)$ is normal to the level surface $F=0$ at (x,y,z).

We want ∇F to be parallel to $\vec{n} = (0,1,1)$.

$\nabla F = (2x, 4y, 8z) = \lambda (0,1,1)$

$$\Rightarrow \left.\begin{array}{l} 2x = 0 \\ 4y = \lambda \\ 8z = \lambda \end{array}\right\} \quad x = 0 \text{ and } y = 2z$$

$$\Downarrow$$

$$0^2 + 2(2z)^2 + 4z^2 = 12$$

$$8z^2 + 4z^2 = 12$$

$$z^2 = 1$$

$$z = \mp 1 \Rightarrow y = \mp 2$$

Points are $(0,2,1)$ and $(0,-2,-1)$

Question 2 (4+4+6=14 points)

Let $f(u)$ be a differentiable function with $f(1) = 1, f'(1) = 2$ and $g(x, y) = xf(\frac{y}{x^2})$.

a) Compute $g_x(1, 1)$.

$$g_x(x,y) = f\left(\frac{y}{x^2}\right) + x\, f'\left(\frac{y}{x^2}\right) y\, (-2) \frac{1}{x^3}$$

$$g_x(1,1) = f(1) + 1\, f'(1)\, 1\, (-2) \frac{1}{1}$$

$$= 1 + 2(-2) = -3$$

b) Find a vector $\vec{v}$ in the xy-plane such that $h(x, y) = 2x^2 - 6xy + 3y^2$ increases most rapidly when (x, y) moves along $\vec{v}$ starting at $(1, 1)$.

$$\operatorname{Max}_{\vec{v}} \left(D_{\vec{v}} h(x,y) \right) = \operatorname{Max}_{\vec{v}} \left(\nabla h \cdot \vec{v} \right) = \operatorname{Max}_{\vec{v}} \left(\|\nabla h\| \cos\theta \right)$$

$$= \|\nabla h\| \operatorname{Max}_{\hat{v}} (\cos\theta) = \|\nabla h\|$$

Hence to get Maximum value of $D_{\vec{v}} h(x,y)$ (which is $\|\nabla h\|$) we must take $\hat{v}$ so that $\cos\theta = 1$ i.e. $\theta = 0$ i.e. $\hat{v} = \frac{\nabla h}{\|\nabla h\|}$

thus $\vec{v} = \frac{\nabla h(1,1)}{\|\nabla h(1,1)\|} = \frac{(-2,0)}{2} = (-1, 0)$

$h_x = 4x - 6y$

$h_y = -6x + 6y$

c) Find $\sin(0.002(\frac{\pi}{2} - 0.01)^2)$ approximately using the tangent plane (differential) approximation at a suitable point.

Hint: Consider the function $f(x, y) = \sin(x(\frac{\pi}{2} - y)^2)$.

Let $f(x,y) = \sin\left(x\left(\frac{\pi}{2} - y\right)^2\right)$. Then $\sin\left(0.002\left(\frac{\pi}{2} - 0.01\right)^2\right) = f(0.002, 0.01)$

$$f_x = \left(\frac{\pi}{2} - y\right)^2 \cos\left(x\left(\frac{\pi}{2} - y\right)^2\right) \Rightarrow f_x(0,0) = \left(\frac{\pi}{2}\right)^2$$

$$f_y = -2x\left(\frac{\pi}{2} - y\right) \cos\left(x\left(\frac{\pi}{2} - y\right)^2\right) \Rightarrow f_y(0,0) = 0$$

Then

$$f(0.002, 0.01) \approx \overbrace{f(0,0)}^{=0} + f_x(0,0)(0.002) + f_y(0,0)(0.01)$$

$$= 0 + \left(\frac{\pi}{2}\right)^2 (0.002) + 0$$

$$= \left(\frac{\pi}{2}\right)^2 (0.002)$$

Question 3 (10 points)

a)Find all critical points of $f(x,y) = 2x^3 - 6xy + 3y^2$.

$f_x = 6x^2 - 6y = 0 \Rightarrow \left.\begin{matrix} x^2 = y \\ x = y \end{matrix}\right\} \quad x = x^2 \Rightarrow x = 0 \text{ or } x = 1$

$f_y = -6x + 6y = 0$

$\left.\begin{matrix} x = 0 \Rightarrow y = 0 \\ x = 1 \Rightarrow y = 1 \end{matrix}\right\} \quad (0,0)$ & $(1,1)$ are the critical points

b)Classify all critical points of $f(x,y) = 2x^3 - 6xy + 3y^2$ as local maximum, local minimum or a saddle point using second derivative test.

$f_{xx} = 12x$

$f_{yy} = 6$

$f_{xy} = -6$

at $(0,0)$; $f_{xx} f_{yy} - f_{xy}^2 = -36 < 0$

$\Rightarrow (0,0)$ saddle point

at $(1,1)$, $f_{xx} f_{yy} - f_{xy}^2 = (12)(6) - (-6)^2$

$= 72 - 36 > 0$

also $f_{xx}(1,1) = 12 > 0$

so $(1,1)$ is a local minimum

Question 4 (8 points) Using Lagrange Multipliers Method, find a point of the surface $xyz^2 = 2$ that is closest to the origin.

Distance of a point (x,y,z) to the origin is $d(x,y,z) = \sqrt{x^2+y^2+z^2}$ and this is minimum if and only if $x^2+y^2+z^2$ is minimum

Thus, we need to find;

minimum value of $f(x,y,z) = x^2+y^2+z^2$

subject to $g(x,y,z) = xyz^2 - 2 = 0$

set $\nabla f = \lambda \nabla g$;

$$\boxed{\begin{array}{l} f_x = 2x = \lambda y z^2 = \lambda g_x \\ f_y = 2y = \lambda x z^2 = \lambda g_y \\ f_z = 2z = \lambda 2xyz = \lambda g_z \\ xyz^2 = 2 \end{array}}$$

- First note that $\lambda \neq 0$ (otherwise $x = y = z = 0$, contradicting 4^th^ equation)
- Also $z \neq 0$ (otherwise $x = 0 = y$)

$2xyz = \lambda y^2 z = \lambda x^2 z \underset{\substack{\lambda \neq 0 \\ z \neq 0}}{\Longrightarrow} x^2 = y^2 \Rightarrow \boxed{x = y}$ ($x = -y$ contradicts 4^th^ equation)

$2z = \lambda 2x^2 z \underset{z \neq 0}{\Longrightarrow} 1 = \lambda x^2 \Rightarrow 2x^2 = (\lambda x^2) z^2 \Rightarrow 2x^2 = z^2$

$\underset{(4)}{\Rightarrow} x^2 \cdot 2x^2 = 2$

$\Rightarrow x^4 = 1 \Rightarrow x = \mp 1$

$\Rightarrow y = \mp 1 \Rightarrow z = \mp\sqrt{2}$

Hence points on $xyz^2 = 2$ that we need to check are

$(1, 1, \sqrt{2}) \rightarrow d = 2$

$(-1, -1, \sqrt{2}) \rightarrow d = 2$

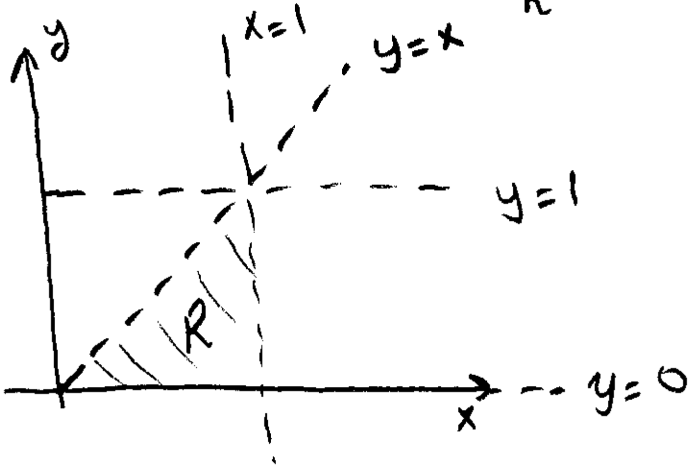
$(1, 1, -\sqrt{2}) \rightarrow d = 2$

$(-1, -1, -\sqrt{2}) \rightarrow d = 2$

These points of the surface $xyz^2 = 2$ are the closest ones to the origin.

Question 5 (10 points)

a)Evaluate $\int_0^1 \int_y^1 \sin(x^2)dxdy$ $= \iint_R \sin(x^2)\,dA = \int_0^1 \int_0^x \sin(x^2)\,dy\,dx$



$$= \int_0^1 y\sin(x^2)\Big|_0^x dx$$

$$= \int_0^1 x\sin(x^2)\,dx$$

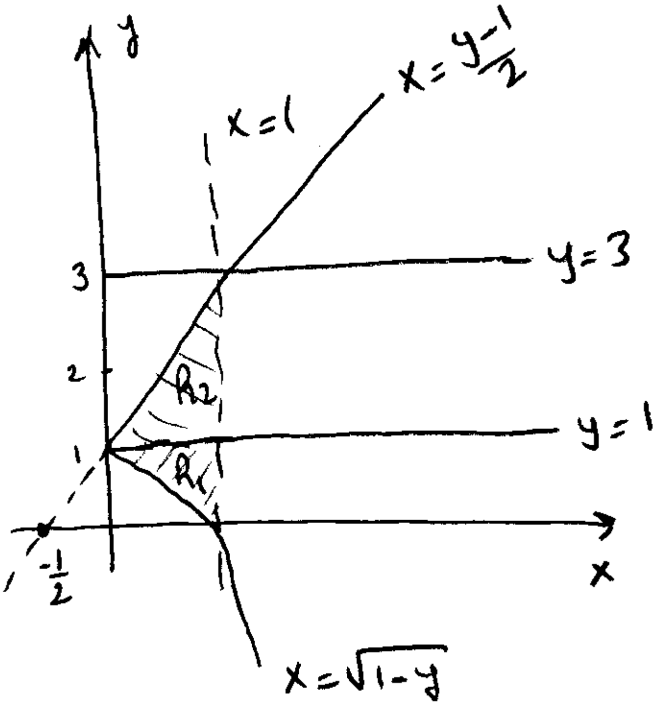
$u = x^2$ $\quad x=0 \Rightarrow u=0$

$\frac{du}{dx} = 2x$ $\quad x=1 \Rightarrow u=1$

$$= \frac{1}{2}\int_0^1 \sin u\,du = -\frac{1}{2}\cos u\Big|_0^1$$

$$= -\frac{1}{2}(\cos 1 - \cos 0) = \frac{1}{2}(1-\cos 1)$$

b)After drawing the region for each integral, express $\int_0^1 \int_{\sqrt{1-y}}^1 f(x,y)dxdy$ + $\int_1^3 \int_{\frac{y-1}{2}}^1 f(x,y)dxdy$ as one double integral.



$$= \iint_{R_1} f(x,y)dxdy + \iint_{R_2} f(x,y)dxdy$$

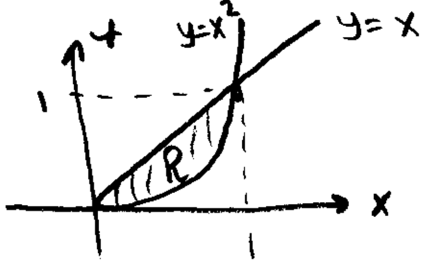
$$= \int_0^1 \int_{1-x^2}^{2x+1} f(x,y)\,dy\,dx$$

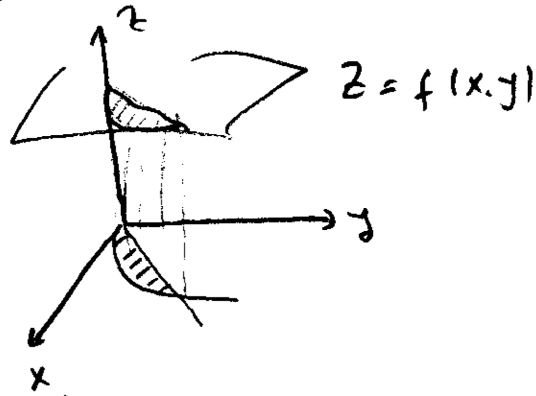
$x = \sqrt{1-y} \Rightarrow x^2 = 1-y \Rightarrow y = 1-x^2$

$x = \frac{y-1}{2} \Rightarrow 2x+1 = y$

Question 6 (10 points)

Let R be the finite region, bounded by the line $y = x$ and the curve $y = x^2$. Find the volume of the solid over R bounded from below by the xy–plane and the graph of $f(x, y) = xy$.





$$V = \iint_R f(x,y)\, dA = \int_0^1 \int_{x^2}^{x} xy \, dy \, dx$$

$$= \frac{1}{2} \int_0^1 xy^2 \Big|_{x^2}^{x} dx = \frac{1}{2} \int_0^1 x\,(x^2 - x^4)\, dx$$

$$= \frac{1}{2} \int_0^1 x^3 - x^5 \, dx = \frac{1}{2} \left[\frac{x^4}{4} - \frac{x^6}{6} \Big|_0^1 \right] = \frac{1}{2}\left(\frac{1}{4} - \frac{1}{6}\right)$$

$$= 1/24$$

M E T U
Department of Mathematics

Group	CALCULUS II Final Exam	List No.

Code : *Math 120* Acad. Year : *2005-2006* Semester : *Spring* Coordinator : *Muhiddin Uguz* Date : *May.25th.2006* Time : *09:30* Duration : *150 minutes*	Last Name : Name : Student No. : Department : Section : Signature : 8 QUESTIONS ON 6 PAGES TOTAL 90 POINTS

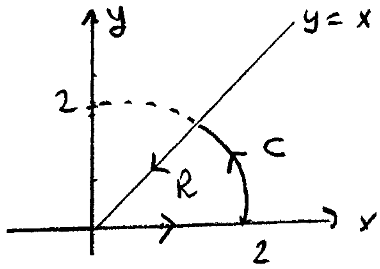
1	2	3	4	5	6	7	8	*SHOW YOUR WORK*	

Question 1 (10 points) Evaluate $\oint_C (-10y^3x^2 + \cos(x^2))\, dx \;+\; (3x^5 + 15y^4x + \sin(y^2))\, dy$ where C is counterclockwise oriented perimeter of the region bounded by $y = x,\ x = 0,\ x^2+y^2 = 4$ in the first quadrant.

Let $P(x,y) = -10y^3x^2 + \cos x^2$
$Q(x,y) = 3x^5 + 15y^4x + \sin y^2$
$\Big\}$ $P_y = -30y^2x^2$, $Q_x = 15x^4 + 15y^4$

- since $P_y \neq Q_x$, (P,Q) is not exact (ie, $\nexists\, f(x,y)$ with $\nabla f = (P,Q)$)
- P & Q are polynomials, hence have all partial derivatives of all order.

let $C = \partial R$ (boundary curve of the region R)



We can use Green's Theorem;

$$\oint_{C=\partial R} P\,dx + Q\,dy = \iint_R (Q_x - P_y)\, dA$$

$$= \iint_R 15x^4 + 15y^4 + 30x^2y^2 \; dA = 15\iint_R (x^2+y^2)^2\, dA$$

$$= 15\int_0^{\pi/4}\int_0^2 r^4 \cdot r \, dr\, d\theta = 15\int_0^{\pi/4} \frac{r^6}{6}\Big|_0^2 d\theta = \frac{15}{6}\, 2^6 \cdot \frac{\pi}{4}$$

$$= 40\pi$$

Question 2 (6+10=16 points)

a)Determine whether the following series is convergent or divergent. If convergent, find the sum $\sum_{k=1}^{\infty} \ln(\frac{\arctan(k+1)}{\arctan(k)})$ $= \sum_{k=1}^{\infty} \left(\ln(\arctan(k+1)) - \ln(\arctan(k))\right)$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \ln(\arctan k+1) - (\ln \arctan k)$$

$$= \lim_{n\to\infty} \left[\left(\ln(\cancel{\arctan 2}) - \ln(\arctan 1)\right) + \left(\ln(\cancel{\arctan 3}) - \ln(\cancel{\arctan 2})\right) + (\ln(\arctan 4) - \ln(\cancel{\arctan 3}) + \cdots + \left(\ln(\arctan n+1) - \ln \cancel{\arctan n}\right) \right.$$

$$= \lim_{n\to\infty} -\ln(\arctan 1) + \ln(\arctan(n+1))$$

$$= \lim_{n\to\infty} \ln\left(\frac{\arctan(n+1)}{\arctan 1}\right) = \lim_{n\to\infty} \ln\left(\frac{\arctan(n+1)}{\pi/4}\right) = \ln\left(\frac{\pi/2}{\pi/4}\right)$$

$= \ln 2$ - convergent

b)Find the radius **R** and interval **I** of convergence of $\sum_{k=1}^{\infty} \frac{1}{k}(x-1)^k$

Let's apply Ratio Test

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \frac{\frac{1}{k+1}|x-1|^{\cancel{k+1}}}{\frac{1}{k}\cancel{|x-1|^k}} = |x-1| \lim_{k\to\infty} \frac{k}{k+1} = |x-1|$$

Hence given power serie around $x_0 = 1$ is convergent (absolutely) if $|x-1| < 1$. That is convergent in $(0,2)$. Thus $\boxed{R = 1}$

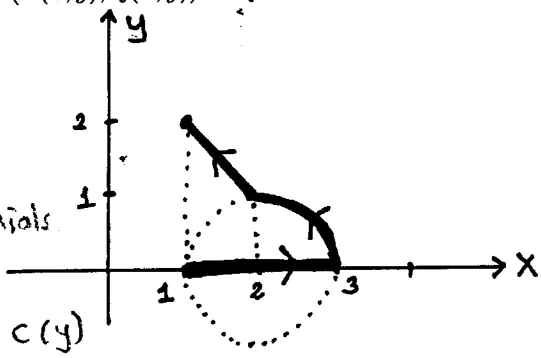
Check the end points:

$x = 0 \Rightarrow \sum_{k=1}^{\infty} \frac{1}{k}(-1)^k$ is convergent alternating series $\left(a_k = \frac{1}{k} > 0, \ a_k \searrow, \ \lim_{n\to\infty} a_k = 0\right)$

$x = 2 \Rightarrow \sum_{k=1}^{\infty} \frac{1}{k}$ is divergent (by p-test)

Thus $I = [0, 2)$

Question 3 (4+4+4=12 points) It is given that $F(x,y) = (P(x,y), Q(x,y))$ where
$P(x,y) = e^x \sin(y) + 2y$,
$Q(x,y) = e^x \cos(y) + 2x - 2y$ and
C_1 is the curve from the point $(1,0)$ to $(1,2)$ as in the figure:



a) Prove that $P\,dx + Q\,dy$ is exact (that is (P,Q) is conservative), by finding a potential function $\phi(x,y)$ such that $\phi_x = P$, $\phi_y = Q$.

$P_y = e^x \cos y + 2 = Q_x$ and P, Q are polynomials so (P,Q) is exact on $\mathbb{R}^2$.

$\phi_x = P = e^x \sin y + 2y \Rightarrow \phi = e^x \sin y + 2yx + c(y)$

$\Rightarrow \phi_y = e^x \cos y + 2x + c'(y) = Q = e^x \cos y + 2x - 2y \Rightarrow c'(y) = -2y$

$\Rightarrow c(y) = -y^2 + c$

$\Rightarrow \phi(x,y) = e^x \sin y + 2yx - y^2 + c$

b) Using ϕ, evaluate $\int_{C_1} P\,dx + Q\,dy$. Since (P,Q) is conservative, the given line integral depends only on the end points of C_1. Hence

$\int_{C_1} P\,dx + Q\,dy = \phi(\text{terminal pt of } C_1) - \phi(\text{initial pt. of } C_1)$

$= \phi(1,2) - \phi(1,0)$

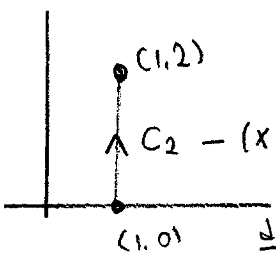
$= e \sin 2$

c) Evaluate $\int_{C_2} P\,dx + Q\,dy$ where C_2 is the line segment from the point $(1,0)$ to the point $(1,2)$.

First way: since (P,Q) is conservative, integral depends only on the end points of C_2 which are the same as of C_1, hence

$\int_{C_2} P\,dx + Q\,dy = \int_{C_1} P\,dx + Q\,dy = e \sin 2$

Second way:



$C_2 - (x(t), y(t)) = (1, 2t)$
$t \in [0,1]$

$\frac{dx}{dt} = 0, \quad \frac{dy}{dt} = 2$

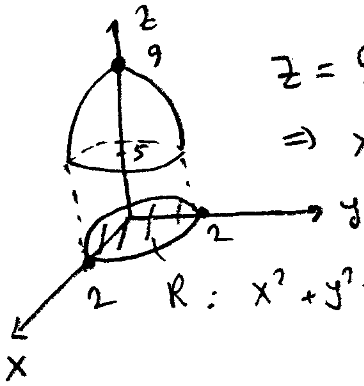
$\int_{C_2} P\,dx + Q\,dy = \int_0^1 (e \sin 2t + 4t)0 + (e \cos 2t + 2 - 4t)2$

$= \int_0^1 2e \cos 2t + 4 - 8t \, dt$

$= 2e \sin \frac{2t}{2} + 4t - \frac{8t^2}{2} \Big|_0^1$

$= e \sin 2$

Question 4 (10 points) Find the volume of the solid bounded by the paraboloid $z = 9 - x^2 - y^2$ and the plane $z = 5$.



$R: x^2 + y^2 = 4$

$z = 9 - x^2 - y^2 = 5$

$\Rightarrow x^2 + y^2 = 4$

$$V = \iint_R (9 - x^2 - y^2 - 5)\, dA$$

$$= \iint_R (4 - r^2)\, r\, dr\, d\theta$$

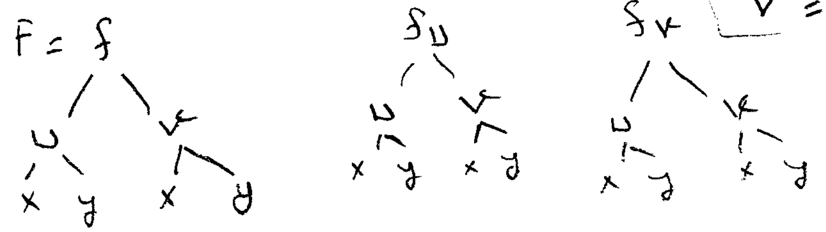
$$= \int_0^{2\pi} \int_0^2 (4r - r^3)\, dr\, d\theta$$

$$= 2\pi \left[4\frac{r^2}{2} - \frac{r^4}{4} \right]_0^2 = 8\pi$$

Question 5 (10 points) Let $F(x, y) = f(x^2 y + x, 3x - y^2)$ where f is a known continuously differentiable function. Find F_{yx} in terms of partial derivatives of f.

$F(x,y) = f(u, v)$ where $u = x^2 y + x \Rightarrow \begin{cases} u_x = 2xy + 1 \\ u_y = x^2 \end{cases}$

$v = 3x - y^2 \Rightarrow \begin{cases} v_x = 3 \\ v_y = -2y \end{cases}$



$F_y = f_u u_y + f_v v_y = f_u x^2 + f_v(-2y)$

$F_{yx} = (f_{uu} u_x + f_{uv} v_x) x^2 + f_u 2x +$

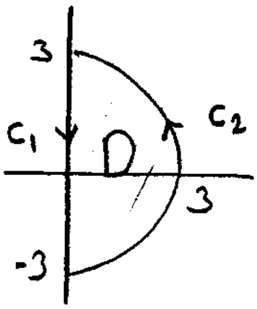
$(f_{vu} u_x + f_{vv} v_x)(-2y)$

$= f_{uu}(2xy+1)x^2 + f_{vv}(3)(-2y) + f_{uv}(3x^2 + (2xy+1)(-2$

$+ f_u \cdot 2x$

Surname: .... Name: .... Id: .... Section: ....

Question 6 (12 points) Find the absolute maximum and the absolute minimum of the function $f(x,y) = x^2 + y^2 - 2x + 4y + 6$ on the domain $D = \{(x,y);\ x^2 + y^2 \le 9,\ \ x \ge 0\}$.



Note that f is a polynomial and D is closed, hence f has absolute Max / min on D. Being a polynomial, f has no singular points. Hence to find abs. Max/min. of f on D, we must check the values of f at critical points inside D and check f on C_1 & on C_2.

inside D • $f_x = 2x - 2 = 0$, $f_y = 2y + 4 = 0$ $\Rightarrow (x,y) = (1,-2)$ is the only critical point inside. and $f(1,-2) = 1 + 4 - 2 - 8 + 6 = 1$

• on C_1 : $(x,y) = (0,t)$, $t \in [-3,+3]$ (orientation is not important!)

$\Rightarrow f\big|_{C_1} = t^2 + 4t + 6$; $t \in [-3,+3]$

$f' = 2t + 4 = 0 \Rightarrow t = -2$ critical pt. : $f\big|_{C_1}(-2) = 2$

end pts: $f\big|_{C_1}(-3) = 3$, $f\big|_{C_1}(3) = 27$

• On C_2 : $\begin{cases} \text{Maximize } f(x,y) = x^2 + y^2 - 2x + 4y + 6 \\ \text{Subject to } g(x,y) = x^2 + y^2 - 9 = 0 \end{cases}$

ie, $\begin{cases} \text{Maximize } f(x,y) = 9 - 2x + 4y + 6 \\ \text{subject to } g(x,y) = x^2 + y^2 - 9 = 0 \end{cases}$

set $\nabla f = \lambda \nabla g \Rightarrow \begin{cases} -\cancel{2} = \cancel{2}\lambda x \\ 2\cancel{4} = \cancel{2}\lambda y \\ x^2 + y^2 = 9 \end{cases}$ $x^2 + y^2 = \frac{1}{\lambda^2} + \frac{4}{\lambda^2} = \frac{5}{\lambda^2} = 9$

$\Rightarrow \lambda = \mp \frac{\sqrt{5}}{3}$

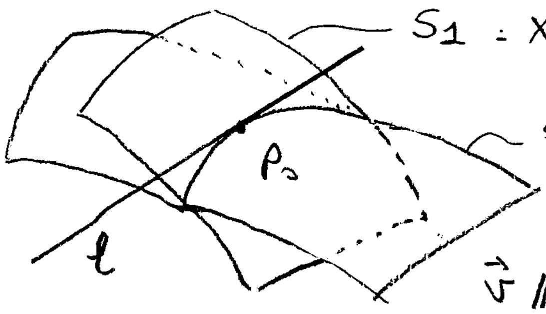
$\Rightarrow x = \frac{-1}{\lambda}$ and $x \geq 0$ so $\lambda < 0$ $\Rightarrow x = 3/\sqrt{5}$

$y = 2/\lambda$ $\Rightarrow y = -\frac{6}{\sqrt{5}}$

$f\left(\frac{3}{\sqrt{5}}, \frac{-6}{\sqrt{5}}\right) = \frac{9}{5} + \frac{36}{5} - \frac{6}{\sqrt{5}} - \frac{24}{\sqrt{5}} + 6 = 15 - \frac{30}{\sqrt{5}}$

Abs. Max $= f(0,3) = 27$, Abs. min $= f(1,-2) = 1$

Question 7 (10 points) Find an equation of the tangent line at $P_0(1,-1,1)$ to the curve of intersection of the surfaces $x^2+y^2=2$ and $y^2+z^2=2$.



$S_1: x^2+y^2=2 \longleftrightarrow f(x,y,z)=x^2+y^2-2=0$

$S_2: y^2+z^2=2 \longleftrightarrow g(x,y,z)=y^2+z^2-2=0$

Direction vector $\vec{v}$ of ℓ is parallel to $\vec{n}_1 \times \vec{n}_2$

$$\vec{v} \parallel \vec{n}_1 \times \vec{n}_2 = \nabla f \times \nabla g \Big|_{P_0} = (2x, 2y, 0) \times (0, 2y, 2z)\Big|$$

$$= (2,-2,0) \times (0,-2,2) \parallel (1,-1,0) \times (0,-1,1)$$

$$= \begin{vmatrix} i & j & k \\ 1 & -1 & 0 \\ 0 & -1 & 1 \end{vmatrix} = (-1,-1,-1) \parallel (1,1,1) = \vec{v}$$

Tangent line ;

$x = 1 + t$
$y = -1 + t$
$z = 1 + t$
$t \in \mathbb{R}$

OR $x-1 = y+1 = z-1 \ (= t)$

OR $\vec{P} = \vec{P_0} + t\vec{v}$

$(x,y,z) = (1+t, -1+t, 1+t)$
$t \in \mathbb{R}$.

Question 8 (10 points) Find $\int_C x^2\, ds$ along the line of intersection of the two planes $x-y+z=0$ and $x+y+2z=0$, from the origin to the point $(3,1,-2)$.

C can be parametrized as $r(t) = (3t, t, -2t)$; $t \in [0,1]$

$r'(t) = (3,1,-2) \Rightarrow \|r'(t)\| = \sqrt{14}$

Thus

$$\int_C x^2\, ds = \int_0^1 9t^2 \cdot \sqrt{14}\, dt = 3\sqrt{14}.$$