M E T U

Department of Mathematics

Group	CALCULUS II Mid Term 1	EXAM PLACE

Code : *Math 120*
Acad. Year : *2006-2007*
Semester : *Spring*
Coordinator: *Muhiddin Uguz*

Date : *April.7.2007*
Time : *09:30*
Duration : *120 minutes*

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Name : Student No. :
Department : Section :
Signature :

8 QUESTIONS ON 6 PAGES
TOTAL 60 POINTS

1	2	3	4	5	6	7	8	SHOW YOUR WORK	

Question 1 (2+2+2+2=8 points) Give an example(no explanations needed) of

- a conditionally convergent series: $\sum \frac{(-1)^n}{n}$, $\sum (-1)^n \frac{\ln(n)}{n}$, ...

- a divergent series $\sum a_n$ where $\{a_n\}$ is a bounded sequence: $\sum 120$, $\sum 5+\frac{1}{n}$, ...

- an alternating divergent series: $\sum (-120)^n$, $\sum (-1)^n$, ...

- an increasing and convergent sequence: $a_n = \{2007 - \frac{1}{n}\}$, ...

Question 2 (10 points) Find the radius of convergence **R** and the interval of convergence **I** (do **not** forget to check the end points)of the power series $\sum_{k=0}^{\infty} \frac{k(-1)^k(x+1)^k}{2^k(k+1)}$

$$a_k = \frac{k\,(-1)^k\,(x+1)^k}{2^k\,(k+1)}$$

Apply Ratio Test;

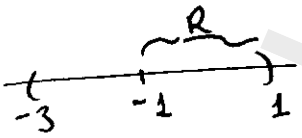
$$\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = \lim_{k\to\infty} \frac{(k+1)\;|x+1|^{\cancel{k+1}}}{2^{\cancel{k+1}}\;(k+2)} \cdot \frac{\cancel{2^k}\,(k+1)}{k\;\cancel{|x+1|^k}} = \frac{|x+1|}{2} \lim_{k\to\infty} \frac{(k+1)^2}{k(k+2)}$$

$$= \frac{|x+1|}{2}$$

Thus, given power series is convergent (absolutely) if $\frac{|x+1|}{2} <$

(and is not absolutely convergent, hence divergent if $\frac{|x+1|}{2} > 1$

$|x+1| < 2 \iff -2 < x+1 < 2 \iff -3 < x < 1$



radius $R = 2$

check the end points;

$x = -3 \Rightarrow \sum a_k = \sum_{k=0}^{\infty} \frac{k\,\cancel{(-1)^k}\,\cancel{(-1)^k}\,\cancel{2^k}}{\cancel{2^k}\,(k+1)} = \sum \boxed{\frac{k}{k+1}} \rightarrow 1 \neq 0$ is divergent by n-th Term Test

$x = 1 \Rightarrow \sum a_k = \sum_{k=0}^{\infty} \frac{k\,(-1)^k\,\cancel{2^k}}{\cancel{2^k}\,(k+1)} = \sum \boxed{(-1)^k \frac{k}{k+1}} \not\rightarrow 0$ is divergent by n^{th} T.T.

Thus $I = (-3, 1)$

Question 3 (7+3=10 points) Let $f(x) = \frac{2x^3}{2-x}$

a)Find the Maclaurin Series (Taylor Series around 0) of $f(x)$. What is the interval of convergence(do **not** forget to check the end points)?

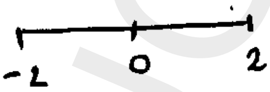
Recall that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \forall\ |x| < 1$

$$f(x) = \frac{2x^3}{2-x} = \cancel{2}x^3 \cdot \frac{1}{\cancel{2}} \frac{1}{1-\frac{x}{2}} = x^3 \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{x^{n+3}}{2^n}$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

Thus $f(x) = \sum_{n=0}^{\infty} \frac{x^{n+3}}{2^n} \quad \forall\ \left|\frac{x}{2}\right| < 1$ ie $|x| < 2$ ✓

$R = 2$



$x = 2 \Rightarrow \sum_{n=0}^{\infty} \frac{2^{n+3}}{2^n} = \sum_{n=0}^{\infty} 8$ divergent (by n^{th} T.T)

$x = -2 \Rightarrow \sum \frac{(-2)^{n+3}}{2^n} = \sum (-1)^{n+3} \frac{\cancel{2^n} \cdot 8}{\cancel{2^n}}$ div. by n^{th} T.T.

$I = (-2, 2)$

b)Find the 2007th derivative $f^{(2007)}(0)$.

$$f(x) = \frac{2x^3}{2-x} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^{n+3}}{2^n}$$

Thus $\frac{f^{(2007)}(0)}{2007!}$ is the coefficient of x^{2007}

so taking $n = 2004$, we get

$$f^{(2007)}(0) = \frac{1}{2^{2004}} \cdot 2007!$$

Question 4 (5 points) Find the sum of the convergent series $\sum_{n=1}^{\infty}\frac{(-1)^n}{e^n\,n}$

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{e^n \cdot n} = \sum_{n=1}^{\infty}\frac{x^n}{n} \quad \text{at} \quad x = \frac{-1}{e}$$

$$\sum_{n=1}^{\infty}\frac{x^n}{n} = \sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1} = \sum_{n=0}^{\infty}\int_0^x t^n\,dt = \int_0^x \underset{\substack{\parallel \\ \frac{1}{1-t}\ \forall |t|<1}}{\boxed{\sum_{n=0}^{\infty} t^n}}\,dt$$

$$= \int_0^x \frac{1}{1-t}\,dt = -\ln(1-t)\Big|_0^x$$

$$= \underset{\searrow 0}{\cancel{\ln(1)}} - \ln(1-x)$$

$$\text{at } x = -\frac{1}{e} \qquad -\ln\left(1+\frac{1}{e}\right) = -\ln\left(\frac{e+1}{e}\right) = \ln\left(\frac{e}{e+1}\right)$$

Question 5 (5 points) Approximate $\frac{1}{e^{\frac{1}{10}}} = e^{-\frac{1}{10}}$ with an error less than $\frac{1}{1000}$

Recall that $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} \quad \forall x$

Thus
$$e^{-\frac{1}{10}} = \sum_{n=0}^{\infty}\frac{\left(\frac{-1}{10}\right)^n}{n!} = \sum_{n=0}^{\infty}\frac{(-1)^n}{10^n \cdot n!}$$

$$= 1 - \frac{1}{10} + \frac{1}{100 \cdot 2} - \frac{1}{1000 \cdot 6} + \cdots$$

$$\approx 1 - \frac{1}{10} + \frac{1}{200} = \frac{181}{200}$$

↑

error $< \frac{1}{6000} < \frac{1}{1000}$

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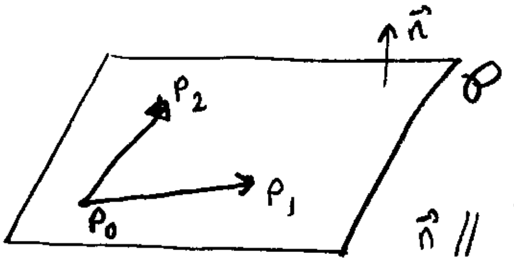
Question 6 (5 points) Find an equation of the plane passing through the points $P_0(1,2,3)$, $P_1(-3,0,5)$ and $P_2(2,-1,0)$.

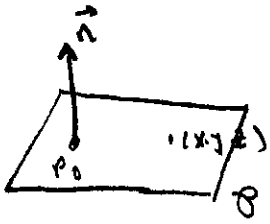


$\overrightarrow{P_0P_1} = (-4,-2,2)$

$\overrightarrow{P_0P_2} = (1,-3,-3)$

$$\vec n \parallel \begin{vmatrix} i & j & k \\ 2 & 1 & -1 \\ -1 & 3 & 3 \end{vmatrix} = i(3+3) - j(6-1) + k(6+1) = (6,-5,7)$$

we can take $\vec n = (6,-5,7)$



$(x,y,z) \in \mathcal{P} \iff (x-1, y-2, z-3) \perp \vec n$

$\iff (x-1, y-2, z-3) \cdot (6,-5,7) = 0$

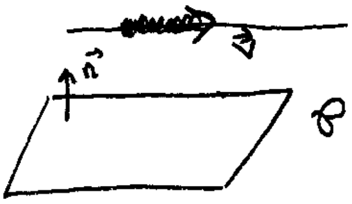
$\iff 6x - 5y + 7z = 17$

Question 7 (5 points) Find the distance between the plane $x + y = 3$ and the line

$x = 119 + t$

$y = 120 - t \quad t \in \Re$. Give explanations.

$z = 2007 + t$



let $\vec v$ be a direction vector of ℓ

and $\vec n$ " " normal " " $\mathcal{P}$

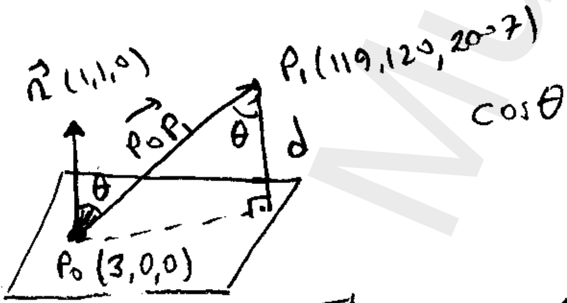
we can take $\vec v = (1,-1,1)$

$\vec n = (1,1,0)$

Then $\vec v \cdot \vec n = 0$ Thus line ℓ is parallel to plane $\mathcal{P}$.

Thus to find the distance, choose ANY point P_1 from ℓ and find $d(P_1, \mathcal{P}) = d(\ell, \mathcal{P})$

let $P_1 = (119, 120, 2007)$. (taking $t=0$).



$$\cos\theta = \frac{d}{\|\overrightarrow{P_0P_1}\|} \Rightarrow d = \|P_0P_1\| \cos\theta = \|P_0P_1\| \frac{\|\vec n\| |\cos\theta|}{\|\vec n\|}$$

$$= \frac{|\overrightarrow{P_0P_1} \cdot \vec n|}{\|\vec n\|}$$

Thus $$d = \frac{|(116, 120, 2007) \cdot (1,1,0)|}{\sqrt 2} = \frac{236}{\sqrt 2}$$

Question 8 (4+4+4=12 points)

Determine whether the followings are convergent or divergent. Give explanations.

a) $\sum_{n=5}^{\infty} \frac{n^2-5n+1}{\sqrt{n}(n^2+1)} = \sum a_n \qquad a_n \geq 0 \quad \forall n = 5, 6, \ldots$

let $b_n = \frac{1}{\sqrt{n}} > 0 \quad \forall n = 5, 6 \ldots$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{x\to\infty} \frac{x^2-5x+1}{\cancel{\sqrt{x}}(x^2+1)} \cdot \cancel{\sqrt{x}} = 1$ – a nonzero finite number.

By limit Comparison Test we have either both $\sum a_n$ & $\sum b_n$ converges or both diverges. Since $\sum b_n = \sum \frac{1}{n^{1/2}}$ is divergent (by p-test) so is $\sum a_n$.

b) $\sum_{n=2}^{\infty} (-1)^n \frac{n^2}{n^3+1} = \sum (-1)^n a_n$

- $a_n > 0$
- $\lim_{n\to\infty} a_n = \lim_{x\to\infty} \frac{x^2}{x^3+1} = \lim_{x\to\infty} \frac{1}{x+\frac{1}{x^2}} = 0$
- let $f(x) = \frac{x^2}{x^3+1}$. Then $f(n) = a_n$. $f'(x) = \frac{2x(x^3+1) - 3x^4}{(x^3+1)^2}$

$$= \frac{-x^4+2x}{(x^3+1)^2} = \overset{\geq 0}{\frac{x}{(x^3+1)^3}} \overset{\leq 0}{(-x^3}$$

$< 0 \quad \forall x \geq 2$

Thus $f(x)$, and hence a_n is decreasing

Therefore $\sum (-1)^n a_n$ is convergent by Alternating Series Test

c) $\{a_n\}_{n=1}^{\infty} = \frac{3^n\, n^2}{n!}$

Consider $\sum a_n = \sum \frac{3^n n^2}{n!}$

Ratio Test:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{3^{\cancel{n+1}} (n+1)^{\cancel{2}}}{\cancel{(n+1)!}} \cdot \frac{\cancel{n!}}{\cancel{3^n} \cdot n^2}$$

$$= \lim_{n\to\infty} \frac{3(n+1)}{n^2} = 0 < 1$$

Thus $\sum a_n$ is conv. by Ratio Test

Hence $\lim_{n\to\infty} a_n = 0$ by n^{th} Term Test

That is $\{a_n\}$ is convergent.

OR

$$0 < a_n = \frac{3^n \cdot n^2}{n!} = \frac{3}{1}\,\frac{3}{2}\,\frac{\cancel{3}}{\cancel{3}} \underbrace{\left(\frac{3}{4} \cdots \frac{3}{n-3}\right)}_{<1} \left(\frac{3n}{n-2}\,\frac{3n}{n-1}\right)\frac{3}{2}$$

$$\leq \frac{9}{2} \cdot \underbrace{\frac{9n^2}{(n-2)(n-3)}}_{\to 9} \cdot \underbrace{\frac{3}{n}}_{\to 0} \to 0$$

$0 \to 0$

Thus by squeezing Thm,

$\lim_{n\to\infty} a_n = 0$

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Group	CALCULUS II Mid Term 2	EXAM PLACE

Code : *Math 120*
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Coordinator: *Muhiddin Uguz*

Date : *May.5.2007*
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6 QUESTIONS ON 6 PAGES
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1	2	3	4	5	6	SHOW YOUR WORK	

Question 1 (5+5=10 points) Find the following limit or show that it does not exist (Show your work):

a) $\lim\limits_{(x,y)\to(0,120)} \frac{\sin(xy)}{x}$ $= \lim\limits_{(x,y)\to(0,120)} \frac{\sin(xy)}{(xy)}(y)$

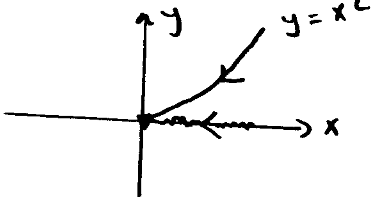
$= \lim\limits_{(x,y)\to(0,120)} \frac{\sin(xy)}{(xy)} \cdot \lim\limits_{(x,y)\to(0,120)} y$

since each limit exists

$u = xy$

$= \lim\limits_{u\to 0} \frac{\sin u}{u} \cdot \lim\limits_{y\to 120} y = 1.120 = 120$

b) $\lim\limits_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$



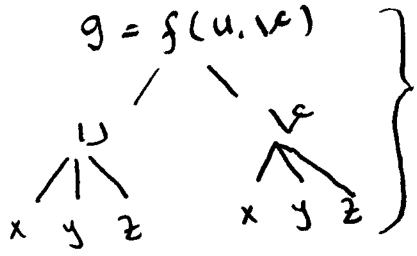
- $\lim\limits_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} = \lim\limits_{x\to 0} \frac{x^2x^2}{x^4+(x^2)^2} = \frac{1}{2}$
  along $y = x^2$
- $\lim\limits_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} = \lim\limits_{\substack{y\to 0\\(y\neq 0)}} \frac{0^2y}{0^4+y^2} = \lim\limits_{y\to 0} 0 = 0$
  along $x = 0$

since $0 \neq \frac{1}{2}$, limit does <u>NOT</u> exist.

Question 2 (5+5=10 points) Let $f(u,v)$ be a function with $f(-1,3)=0$, $f_1(-1,3)=2$ and $f_2(-1,3)=-3$.

a)Let $g(x,y,z)=f(xyz\ ,\ x^2+y^2+z^2)$. Find the gradiant vector $\nabla g(1,-1,1)$.

$$\nabla g(x,y,z) = \langle g_x(x,y,z)\ ,\ g_y(x,y,z)\ ,\ g_z(x,y,z)\rangle$$



$u_x = yz \rightarrow u_x(1,-1,1) = -1$

$u_y = xz \rightarrow u_y(1,-1,1) = 1$

$u_z = xy \rightarrow u_z(1,-1,1) = -1$

$v_x = 2x \rightarrow v_x(1,-1,1) = 2$

$v_y = 2y \rightarrow v_y(1,-1,1) = -2$

$v_z = 2z \rightarrow v_z(1,-1,1) = 2$

$$g_x = f_1(u,v)\, u_x(x,y,z) + f_2(u,v)\, v_x(x,y,z)$$

$$= 2.(-1) + (-3)(2) = -2-6 = -8$$

$$g_y = f_1 u_y + f_2 v_y = (2)(1) + (-3)(-2) =$$

$$= 2+6 = 8$$

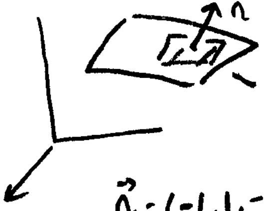
$$g_z = f_1 u_z + f_2 v_z = (2)(-1) + (-3)(2)$$

$$= -2-6 = -8$$

Thus

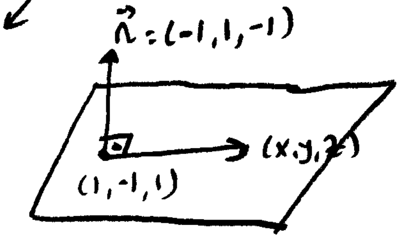
$$\boxed{\nabla g(1,-1,1) = \langle -8, 8, -8\rangle}$$

b)Find an equation of the tangent plane to the surface $f(xyz\ ,\ x^2+y^2+z^2)=0$ at the point $(x,y,z)=(1,-1,1)$.



$\vec{n} \parallel \nabla g(1,-1,1) = \langle -8, 8, -8\rangle$

$g(x,y,z) = f(u,v) = 0$



$(x,y,z) \in$ Tangent plane $\Leftrightarrow$

$$(x-1, y+1, z-1) \bullet (-1,1,-1) = 0$$

$$\boxed{-x+y-z = -3}$$

Question 3 (10 points) Find and classify ALL critical points of $f(x, y) = x^2y + xy^2 + 3xy$.

$$\begin{matrix} f_x = 2xy + y^2 + 3y = 0 \\ f_y = x^2 + 2xy + 3x \;/\!/ \end{matrix} \Rightarrow \left[\begin{matrix} y(2x + y + 3) = 0 \\ x\;(2y + x + 3) = 0 \end{matrix}\right.$$

$$\Rightarrow \begin{cases} \to y = 0 \Rightarrow x(x+3) = 0 \Rightarrow \begin{cases} x = 0 \Rightarrow \boxed{(x,y) = (0,0)} \\ x = -3 \Rightarrow \boxed{(x,y) = (-3,0)} \end{cases} \\ \to y \neq 0 \Rightarrow 2x + y + 3 = 0 \Rightarrow y = -2x - 3 \Rightarrow \end{cases}$$

$$\Rightarrow x(-4x - 6 + x + 3) = 0 \Rightarrow x(-3x - 3) = 0$$

$$\Rightarrow \begin{cases} x = 0 \Rightarrow y = -3 \Rightarrow \boxed{(x,y) = (0,-3)} \\ x \neq 0 \Rightarrow -3x - 3 = 0 \\ \quad \Rightarrow x = -1 \Rightarrow y = -1 \\ \quad \Rightarrow \boxed{(x,y) = (-1,-1)} \end{cases}$$

The set of ALL critical points is

$\{(0,0), (-3,0), (0,-3), (-1,-1)\}$

To classify them;

$$\left.\begin{matrix} f_{xx} = 2y \\ f_{xy} = 2x + 2y + 3 \\ f_{yy} = 2x \end{matrix}\right\} \quad \Delta = f_{xx} f_{yy} - f_{xy}^2$$

at $(0,0)$: $\Delta = 0 - 9 < 0 \Rightarrow (0,0)$ is a saddle point

at $(0,-3)$: $\Delta = 0 - 9 < 0 \Rightarrow (0,-3)$ - - - - -

at $(-3,0)$: $\Delta = 0 - 9 < 0 \Rightarrow (-3,0)$ - - - - -

at $(-1,-1)$: $\Delta = 4 - 1 = 3 > 0$ & $f_{xx} = -2 < 0 \Rightarrow$

$(-1,-1)$ is a local MAX.

Question 4 (10 points) Find the MAXIMUM and the minimum values of $f(x,y,z) = x^2 + yz - 5$ on the sphere $x^2 + y^2 + z^2 = 1$.

Maxmize/minimize $f(x,y,z) = x^2+yz-5$

subject to $g(x,y,z) = x^2+y^2+z^2-1 = 0$

Use Lagrange multipliers Method;

set $\nabla f = \lambda \nabla g$

$$\left.\begin{array}{l} f_x = \lambda g_x \\ f_y = \lambda g_y \\ f_z = \lambda g_z \end{array}\right\} \quad \left.\begin{array}{l} 2x \overset{①}{=} 2\lambda x \\ z \overset{②}{=} 2\lambda y \\ y \overset{③}{=} 2\lambda z \\ x^2+y^2+z^2-1 \overset{④}{=} 0 \end{array}\right]$$

4 equations
4 unknowns ✓

① $\Rightarrow 2x(1-\lambda) = 0 \rightarrow x = 0$ ($\Rightarrow$ ②⊕④ $\Rightarrow y \neq 0$ & $z \neq 0$)

$\overset{②}{\Rightarrow} \lambda = \frac{z}{2y}$ & $\overset{③}{\Rightarrow} \lambda = \frac{y}{2z}$

$\Rightarrow \frac{z}{2y} = \frac{y}{2z} \Rightarrow y^2 = z^2 \Rightarrow y = \mp z$

$\overset{④}{\Rightarrow} 2y^2 = 1 \Rightarrow y = \mp \frac{1}{\sqrt{2}}$

$\Rightarrow (x,y,z) = (0, \frac{1}{\sqrt{2}}, \mp\frac{1}{\sqrt{2}})$ or $(0, \frac{-1}{\sqrt{2}}, \mp\frac{1}{\sqrt{2}})$

$x \neq 0 \Rightarrow \lambda = 1 \overset{②}{\Rightarrow} z = 2y \overset{③}{=} 4z \Rightarrow z = 0 = y$

$\Downarrow$

$x = \mp 1$

$\Rightarrow (x,y,z) = (\mp 1, 0, 0)$

$f(0, \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}) = f(0, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = \frac{-1}{2} - 5 = -\frac{11}{2}$

$f(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = f(0, \frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}) = \frac{1}{2} - 5 = -\frac{9}{2}$

$f(\mp 1, 0, 0) = 1 - 5 = -4$

$\boxed{\text{MAX} = -4}$ $\boxed{\text{min} = -\frac{11}{2}}$

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Question 5 (10 points) Let $f(x,y)$ be a function which has directional derivative in any direction at (a,b). It is also given that for $\vec{u} = \frac{1}{\sqrt{2}}\vec{i} - \frac{1}{\sqrt{2}}\vec{j}$ and $\vec{v} = \frac{1}{2}\vec{i} + \frac{\sqrt{3}}{2}\vec{j}$, the directional derivatives are $(D_{\vec{u}}f)(a,b) = \frac{1}{\sqrt{2}}$ and $(D_{\vec{v}}f)(a,b) = 2$. What is the MAXIMUM rate of increase of f at (a,b)?

$$D_{\vec{w}} f(a,b) = \nabla f(a,b) \cdot \vec{w} = \|\nabla f(a,b)\| \, \overset{\overset{1}{\parallel}}{\overbrace{\|\vec{w}\|}} \, \underset{\underset{-1}{\le}}{\overset{\le 1}{\cos\theta}}$$

it's MAXIMUM value is $\|\nabla f(a,b)\|. = \sqrt{A^2+B^2}$ where

$A = f_x(a,b)$, $B = f_y(a,b)$

- $D_{\vec{u}} f(a,b) = (A,B) \cdot \vec{u}$

$$\Rightarrow \frac{1}{\sqrt{2}} = (A,B) \cdot \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right) = \frac{A-B}{\sqrt{2}}$$

$$\Rightarrow \boxed{A - B = 1}$$

- $D_{\vec{v}} f(a,b) = (A,B) \cdot \hat{v}$

$$\Rightarrow 2 = \frac{A}{2} + \frac{\sqrt{3}B}{2} \Rightarrow \boxed{4 = A + \sqrt{3}B}$$

$$\begin{aligned} -A + B &= -1 \\ + \quad A + \sqrt{3}B &= 4 \\ \hline (\sqrt{3}+1)B &= 3 \end{aligned} \Rightarrow B = \frac{3}{\sqrt{3}+1} \quad \Rightarrow A = 1 + B = \frac{4+\sqrt{3}}{\sqrt{3}+1}$$

Thus $\nabla f(a,b) = \left(\frac{4+\sqrt{3}}{\sqrt{3}+1}, \frac{3}{\sqrt{3}+1}\right) = \frac{1}{\sqrt{3}+1}(4+\sqrt{3}, 3)$

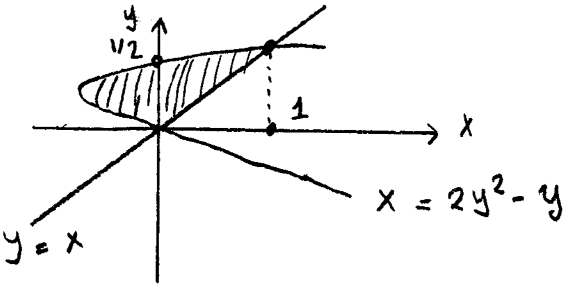
Hence, the maximum value of directional derivative is

$$\|\nabla f(a,b)\| = \frac{1}{\sqrt{3}+1}\left((4+\sqrt{3})^2 + 3^2\right)^{1/2}$$

$$= \frac{1}{\sqrt{3}+1}\left[28 + 8\sqrt{3}\right]^{1/2}$$

Question 6 (5+5=10 points)

a)Evaluate $\int\int_R 2xy \; dA$ where R is the finite region in the xy-plane bounded by the parabola $x = 2y^2 - y$ and the line $y = x$.



$y = 2y^2 - y \Rightarrow 2y^2 - 2y = 0$

$2y(y-1) = 0$

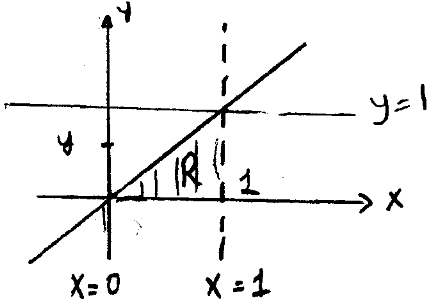
$y = 0 \, , \; y = 1$

$$\int_0^1 \int_{2y^2-y}^{y} 2xy \, dx \, dy = \int_0^1 x^2 y \Big|_{2y^2-y}^{y} dy = \int_0^1 [y^2 - (2y^2-y)^2] \, y \, dy$$

$$= \int_0^1 (-4y^4 + 4y^3) \, y \, dy = \int_0^1 (-4y^5 + 4y^4) \, dy$$

$$= -4\frac{y^6}{6} + 4\frac{y^5}{5} \Big|_0^1 = -\frac{2}{3} + \frac{4}{5} = \frac{2}{15}$$

b)Evaluate $\int_0^1 \int_y^1 2e^{x^2} \, dx \, dy$ $= \iint_R 2e^{x^2} dx \, dy = \int_0^1 \int_0^x 2e^{x^2} dy \, dx$



$$= \int_0^1 2xe^{x^2} dx = e^{x^2} \Big|_0^1$$

$$= e - 1$$

M E T U
Department of Mathematics

Group

CALCULUS II
Final Exam

EXAM PLACE

Code : *Math 120*
Acad. Year : *2006-2007*
Semester : *Spring*
Coordinator: *Muhiddin Uguz*

Date : *May.31.2007*
Time : *09:30*
Duration : *1~~2~~50 minutes*

Last Name :
Name :
Student No. :
Department :
Section :
Signature :

7 QUESTIONS ON 6 PAGES
TOTAL 90 POINTS

1	2	3	4	5	6	7	**SHOW YOUR WORK**	

Question 1 (10+5=15 points) Let $F(x) = \int_0^x e^{-t^2}\,dt$.

a)Find the Maclorian Series (i.e. Taylor Series about 0) of $F(x)$. (You may use Maclorian Series of well-known functions)

Recall that $e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} \;\forall t \in \mathbb{R}$, and hence $e^{-t^2} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!}$.

$$\int_0^x e^{-t^2}\,dt = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)\,n!} = F(x) \quad \forall x \in \mathbb{R}.$$

b) Approximate $F(\frac{1}{2})$ with error less than 0.00001.

$$F(\tfrac{1}{2}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+1}(2n+1)\,n!} \quad \text{is a convergent alternating series}$$

$$\approx \sum_{n=0}^{N} \frac{(-1)^n}{2^{2n+1}(2n+1)\,n!}$$

$$\text{error} = e_N < \frac{1}{2^{2(N+1)+1}\,(2(N+1)+1)\,(N+1)!} < 0.00001$$

$$\Rightarrow 2^{2N+3}(2N+3)(N+1)! > 100{,}000 \Rightarrow N \geq 3$$

Thus

$$F(\tfrac{1}{2}) \approx \sum_{n=0}^{3} \frac{(-1)^n}{2^{n+1}(2n+1)\,n!} = \frac{1}{2} - \frac{1}{24} + \frac{1}{320} - \frac{1}{5376}$$

[redacted]

Question 2 (15 points) Find the minimum and the maximum values of $f(x,y) = 3x^2 + 3y^2 + 2xy + 1$ on the closed disk $x^2 + y^2 \leq 1$.

Interior points

$f_x = 6x + 2y = 0 \Rightarrow y = -3x$ and $x = -3y \Rightarrow (0,0)$ is the

$f_y = 6y + 2x$ "

only critical point and

$\boxed{f(0,0) = 1}$

Boundary points

Maximize/minimize $f(x,y) = 3x^2 + 3y^2 + 2xy + 1$

subject to $g(x,y) = x^2 + y^2 - 1 = 0$

use Lagrange multipliers Method; set $\nabla f = \lambda \nabla g$

$$\begin{array}{ll} f_x = & \boxed{6x + 2y \overset{①}{=} 2\lambda x} = \lambda g_x \\ f_y = & \boxed{6y + 2x \overset{②}{=} 2\lambda y} = \lambda g_y \\ & \boxed{x^2 + y^2 \overset{③}{=} 1} \end{array}$$

3 unknowns, (x, y, λ)
3 Equations.

$x = 0 \overset{①}{\Rightarrow} y = 0 \overset{③}{\Rightarrow} 0 = 1$ a contradiction. $\Rightarrow x \neq 0$

similarly

$y = 0 \overset{②}{\Rightarrow} x = 0 \overset{③}{\Rightarrow} 0 = 1$ a contradiction $\Rightarrow y \neq 0$

$\left.\begin{array}{l} x \neq 0 \\ y \neq 0 \end{array}\right\} \overset{\Rightarrow}{(1,2)} \lambda = \dfrac{3x + y}{x} = \dfrac{3y + x}{y} \Rightarrow \cancel{3xy} + y^2 = \cancel{3xy} + x^2$

$\Rightarrow x^2 = y^2 \overset{③}{\Longrightarrow} x^2 = y^2 = \frac{1}{2}$

$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = f\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right) = \frac{3}{2} + \frac{3}{2} + 1 + 1 = \boxed{5}$

$f\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right) = f\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{3}{2} + \frac{3}{2} - 1 + 1 = \boxed{3}$

Since $f(x,y)$ is continuous and the region $x^2 + y^2 \leq 1$ is closed, f has Max and min values on the region.

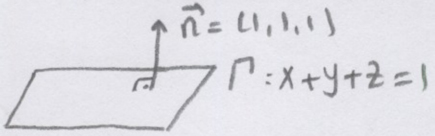
Maximum Value is, being the biggest among the candidates, 5

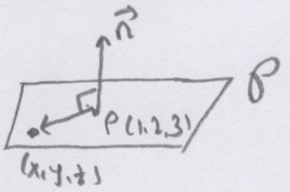
minimum .. .. . . smallest . . . - - , 1

Question 3 (5+5+5=15 points)

Let P be the point $(1, 2, 3)$ and Γ be the plane $x + y + z = 1$.

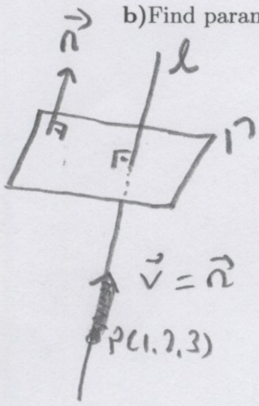
a) Find an equation of the plane through P, parallel to Γ.





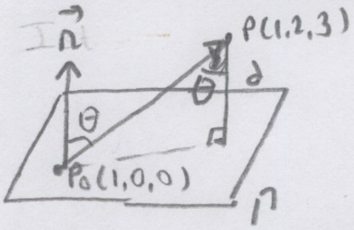
$$(x,y,z) \in \mathcal{P} \iff (x-1, y-2, z-3) \perp \vec{n}$$
$$\iff (x-1, y-2, z-3) \cdot \vec{n} = 0$$
$$\iff x+y+z=6$$

b) Find parametric equation of the line L passing through P and perpendicular to Γ



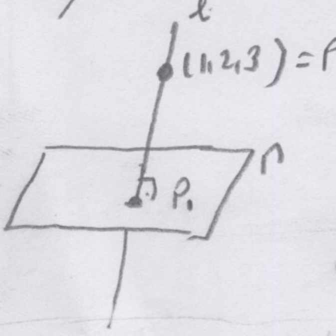
$x = 1 + t$
$y = 2 + t$
$z = 3 + t$
$t \in \mathbb{R}$

c) Find the distance between P and Γ. let P_0 be any point, say $P_0(1,0,0) \in \Gamma$



$$|\cos\theta| = \frac{d}{\|P_0P\|} \Rightarrow d = \|P_0P\| \, |\cos\theta|$$
$$= \frac{\|P_0P\| \, \|\vec{n}\| \cos\theta}{\|\vec{n}\|}$$
$$= \frac{|\overrightarrow{P_0P} \cdot \vec{n}|}{\|\vec{n}\|} = \frac{|(0,2,3) \cdot (1,1,1)|}{\sqrt{3}} =$$

OR/ Use part (b) to find $P_1 = l \cap \Gamma$



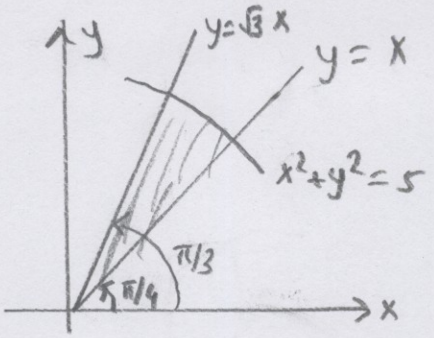
$$\underbrace{1}_{} = x+y+z = 1+t+2+t+3+t = \underbrace{6+3t}_{}$$
$$\Rightarrow t = -5/3 \Rightarrow P_1 = \left(-\frac{5}{3}+1, -\frac{5}{3}+2, -\frac{5}{3}+3\right) = \left(-\frac{2}{3}, \frac{1}{3},\right.$$
$$d = d(P, \Gamma) = d(P, P_1) = \sqrt{\left(1+\frac{2}{3}\right)^2 + \left(2-\frac{1}{3}\right)^2 + \left(3-\frac{4}{3}\right)^2}$$
$$\boxed{= 5/\sqrt{3}}$$

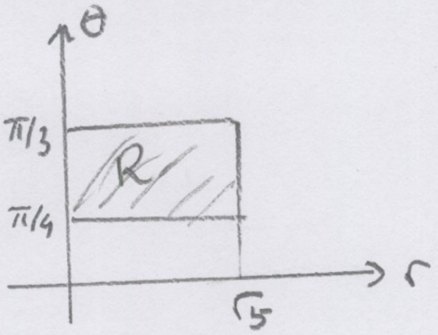
OR/ use part (a) -

$d = d(P, \Gamma) = d(\mathcal{P}, \Gamma) = d(P_2, \Gamma)$ where P_2 is any point, say $P_2 = (6,0,0)$ on $\mathcal{P}$

$= \sqrt{}$

Question 4 (10 points) Evaluate $\int_R\int e^{x^2+y^2}\, dA$ where R is the region in the first quadrant bounded by $y = x$, $y = \sqrt{3}x$, and $x^2 + y^2 = 5$.

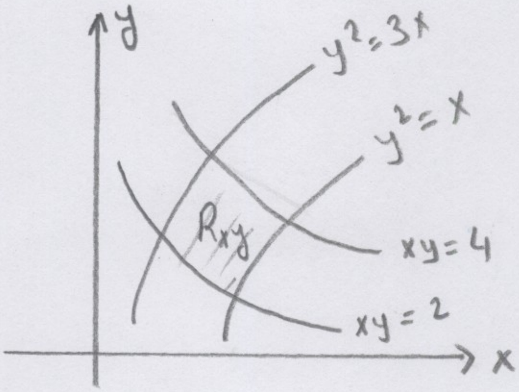




$$\iint_R e^{x^2+y^2}\,dA = \int_{\pi/4}^{\pi/3}\int_0^{\sqrt{5}} e^{r^2}\cdot r\,dr\,d\theta = \left(\frac{\pi}{3}-\frac{\pi}{4}\right)\frac{1}{2}e^{r^2}\Big|_0^{\sqrt{5}}$$

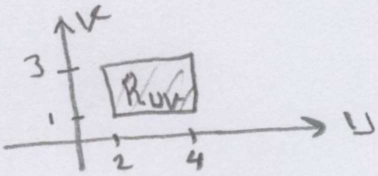
$$= \frac{\pi}{24}\left(e^5-1\right)$$

Question 5 (10 points) Evaluate the double integral $\int_R\int \frac{x^2}{y^4}\, dxdy$ where R is the region bounded by the hyperbolas $xy = 2$, $xy = 4$ and the parabolas $y^2 = x$, $y^2 = 3x$.



let $u = xy$, $v = \frac{y^2}{x}$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} y & x \\ \frac{-y^2}{x^2} & \frac{2y}{x} \end{vmatrix} = \frac{3y^2}{x} = 3v$$



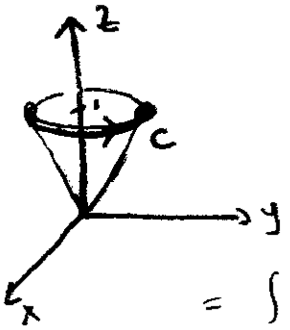
$$\iint_{R_{xy}} \frac{x^2}{y^4}\,dx\,dy = \iint_{R_{uv}} \frac{1}{v^2}\cdot\frac{1}{3v}\,du\,dv = \frac{1}{3}\int_2^4\int_1^3 \frac{1}{v^3}\,dv\,du = 8/27$$

$$= \frac{2}{3}\,\frac{-1}{2v^2}\Big|_1^3 = \frac{8}{27}$$

Surname: .... **Name:** .... **Id:** .... **Section:** ....

Question 6 (5+5+5=15 points) Given $\vec{F}(x,y,z) = xz\vec{i} + (z^2+4)\vec{j} + (x^2+2yz)\vec{k}$ and $\vec{G}(x,y,z) = 2xz\vec{i} + (z^2+4)\vec{j} + (x^2+2yz)\vec{k}$. Let C be the part of the circle of intersection of the cone $z = \sqrt{x^2+y^2}$ and the plain $z = 1$, from $(1,0,1)$ to $(0,1,1)$ in counterclockwise direction when viewed from above.

a)Find $\int_C \vec{F} \cdot d\vec{r}$



$C \longleftrightarrow r(t) = (\cos t, \sin t, 1)$, $t \in [0, \pi/2]$

$dx = -\sin t \, dt$ $\quad dy = \cos t \, dt$, $\quad dz = 0$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{\pi/2} (\cos t(-\sin t) + 5\cos t)\, dt$$

$$= \int_0^{\pi/2} (-\tfrac{1}{2}\sin 2t + 5\cos t)\, dt = \tfrac{1}{4}\cos 2t + 5\sin t \Big|_0^{\pi/2}$$

$$= -\tfrac{1}{4} + 5 - (\tfrac{1}{4} + 0) = 5 - \tfrac{1}{2} = \tfrac{9}{2}$$

b)Show that $\vec{G}$ is conservative by finding a function $f(x,y,z)$ such that $\nabla f = \vec{G}$

$f_x = 2xz \Rightarrow f = x^2 z + c(y,z) \Rightarrow f_y = c_y = z^2 + 4$

$\Rightarrow c = (z^2+4)y + c_1(z)$

$\Rightarrow f = x^2 z + (z^2+4)y + c_1(z)$

$\Rightarrow f_z = x^2 + 2zy + c_1' = x^2 + 2yz \Rightarrow c_1' = 0 \Rightarrow c_1 =$

$\Rightarrow f(x,y,z) = x^2 z + (z^2+4)y + c$

c)Find $\int_C \vec{G} \cdot d\vec{r}$

$$\int_C \vec{G} \cdot d\vec{r} = f(0,1,1) - f(1,0,1) = 5 - 1 = 4$$

Question 7 (10 points) Find $\oint_C 2xy\ dx\ +\ (x^2 + x + \sin(y^2))\ dy$ where C is the circle $x^2 + y^2 = 120$ in counterclockwise direction.

$$\oint_C 2xy\,dx + (x^2+x+\sin(y^2))\,dy \overset{\text{Green's Thm}}{=}$$

$$= \iint_{x^2+y^2\le 120} (2x+1 - 2x)\,dA = \iint_{x^2+y^2\le 120} 1\,dA = \text{Area}$$

$$= \underline{\underline{120\pi}}$$