M E T U

Spring 2009

1956

Math 120- Calculus for Functions of Several Variables
Exam 1 (120 minutes)

1	2	3	4	5	6	7	T

Name :
Lastname :

NO PARTIAL CREDIT WILL BE GIVEN FOR UNSUPPORTED ANSWERS!

---

1. (12 pts) Check if the sequence $\{a_n\}$ is convergent. Explain your answer.

(a) $a_n = (-1)^n \dfrac{\cos n}{n+4}$.

$0 \le |a_n| = \left| (-1)^n \dfrac{\cos n}{n+4} \right| \le \dfrac{1}{n+4}$

Since $\lim\limits_{n\to\infty} \dfrac{1}{n+4} = 0$, by the ST we have $\lim\limits_{n\to\infty} |a_n| = 0$.

But, we know that $\lim\limits_{n\to\infty} a_n = 0$ iff $\lim\limits_{n\to\infty} |a_n| = 0$.

Therefore, $\lim\limits_{n\to\infty} a_n = 0$. The sequence $\{a_n\}$ is convergent.

(b) $a_{n+1} = \dfrac{1}{2}(1 + a_n^2)$, $a_1 = 0$.

$a_1 = \frac{1}{2}(1+0) = \frac{1}{2} < 1$

Suppose that $a_n < 1$. Then $a_{n+1} = \frac{1}{2}(1+a_n^2) < \frac{1}{2}(1+1) = 1$

Therefore, $a_n < 1 \quad \forall n \geq 1$.

Now, $a_{n+1} - a_n = \frac{1}{2}(1+a_n^2 - 2a_n) = \frac{1}{2}(1-a_n)^2 > 0$. So $\{a_n\}$ is increasing.

Since $\{a_n\}$ is increasing and bounded from above, it is convergent.

2. (15 pts) Test for convergence:

(a) $\sum_{n=1}^{\infty} \frac{n+2}{(n+1)^3}$

$$\frac{\frac{n+2}{(n+1)^3}}{\frac{1}{n^2}} = \frac{n^3+2n^2}{(n+1)^3} \longrightarrow 1 \quad \text{as} \quad n \to \infty$$

$\underbrace{\sum \frac{1}{n^2}}_{\substack{p=2>1 \\ p\text{-test}}}$ is convergent. By the LCT, the given series is convergent.

(b) $\sum_{n=1}^{\infty} \frac{n^n}{(3n+2)^n}$

$$\sqrt[n]{a_n} = \frac{n}{3n+2} \longrightarrow \frac{1}{3} \quad \text{as} \quad n \to \infty$$

Since $\frac{1}{3} < 1$, the series is convergent.

(c) $\sum_{n=1}^{\infty} \cos\left(\frac{1}{3^n}\right)$

$$\lim_{n\to\infty} \cos\left(\frac{1}{3^n}\right) = 1 \neq 0$$

By the n-th term test, the series is divergent.

3. (10 pts) Consider the curve $C: \ \vec{r} = < t, \sin t - \cos t, \sin t + \cos t >$, $t \in \mathbb{R}$.

(a) Find the arc length of C from $A = (0, -1, 1)$ to $B = (\pi, 1, -1)$.

$$\vec{r}'(t) = < 1, \cos t + \sin t, \cos t - \sin t >$$

$$|\vec{r}'(t)| = \sqrt{1^2 + (\cos^2 t + \sin^2 t + 2\cancel{\cos t \sin t}) + (\cos^2 t + \sin^2 t - 2\cancel{\cos t \sin t})}$$

$$= \sqrt{3}$$

$$L = \int_0^{\pi} \sqrt{3}\, dt = \sqrt{3}\,\pi$$

(b) Write down an equation of the line tangent to the curve at A.

$$\vec{r}'(0) = < 1, 1, 1 > = \vec{u}$$

$$\ell: \ \frac{x-0}{1} = \frac{y+1}{1} = \frac{z-1}{1} \quad \text{or} \quad \begin{array}{l} x = t \\ y = t-1 \\ z = 1+t \end{array}, \ t \in \mathbb{R}$$

4. (4+8+3 pts) Consider the power series $\sum\limits_{n=1}^{\infty} \underbrace{(-1)^{n-1}\frac{(x-5)^n}{3^n\sqrt{n}}}_{u_n}$.

(a) Find the radius of convergence of the series.

$$\left|\frac{u_{n+1}}{u_n}\right| = \frac{|x-5|}{3}\,\frac{\sqrt{n+1}}{\sqrt{n}} \longrightarrow \frac{|x-5|}{3} \quad \text{as } n\to\infty$$

By the ratio test, the series is convergent if $\frac{|x-5|}{3} < 1$ and is divergent if $\frac{|x-5|}{3} > 1$.

Therefore, the radius of of conv. is 3

(b) Find the interval of convergence of the series.

Endpoints: $|x-5| = 3 \iff x = 8$ & $x = 2$

If $x = 2$, then we have $-\sum\limits_{n=1}^{\infty}\frac{1}{\sqrt{n}} = -\sum\limits_{n=1}^{\infty}\frac{1}{n^{1/2}}$.

Since $p = 1/2 < 1$, the series is divergent.

Let $x = 8$. Then we have $\sum\limits_{n=1}^{\infty}(-1)^{n-1}\frac{1}{\sqrt{n}} = \sum\limits_{n=1}^{\infty}(-1)^{n-1}a_n$

$a_n = \frac{1}{\sqrt{n}} \to 0$ as $n\to\infty$, $a_n > 0$

$a_{n+1} - a_n = \frac{1}{\sqrt{n+1}} - \frac{1}{\sqrt{n}} = \frac{\sqrt{n}-\sqrt{n+1}}{\sqrt{n(n+1)}} < 0 \Rightarrow \{a_n\}$ is decreasing.

By the AST, $\sum\limits_{n=1}^{\infty}(-1)^{n-1}\frac{1}{\sqrt{n}}$ is convergent.

Therefore, the interval of convergence is $(2, 8]$.

(c) Let $f(x)$ denote the sum of the power series given above. What is $f^{(49)}(5)$.

$$f(x) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{3^n\sqrt{n}}(x-5)^n = \sum_{n=1}^{\infty}\frac{f^{(n)}(5)}{n!}(x-5)^n$$

$$\frac{f^{(49)}(5)}{49!} = \frac{1}{3^{49}\sqrt{49}} \Longrightarrow f^{(49)}(5) = \frac{49!}{3^{49}\,7}.$$

5. (15 pts) (a) Find the Taylor series of $f(x) = \dfrac{8}{8+x^3}$ about $a = 0$. (Hint: Use a geometric series.)

$$f(x) = \frac{8}{8\left(1+\left(\frac{x}{2}\right)^3\right)} = \frac{1}{1+\left(\frac{x}{2}\right)^3}$$

We know that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x|<1.$$

Hence,

$$\frac{1}{1-\left(-\frac{x}{2}\right)^3} = \frac{1}{1+\left(\frac{x}{2}\right)^3} = \sum_{n=0}^{\infty} (-1)^n\left[\left(\frac{x}{2}\right)^3\right]^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{3n}} x^{3n}, \quad \underbrace{\left|\frac{x}{2}\right|<1}_{|x|<2} \quad (*)$$

(b) Use part (a) to find $\displaystyle\int_0^1 \frac{8}{8+x^3}\,dx$ as a series.

We integrate the series term-by-term from 0 to 1.
and note that $[0,1] \subseteq (-2,2)$:

So $$\int_0^1 \frac{1}{1+\left(\frac{x}{2}\right)^3}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{3n}} \frac{1}{3n+1}$$

i.e.,
$$\int_0^1 \frac{8}{8+x^3}\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2^{3n}(3n+1)}$$

(c) What can you say about the error if the integral is approximated by taking the first three terms of the series? Explain.

$$\int_0^1 \frac{8}{8+x^3}\,dx = 1 - \frac{1}{2^3\cdot 4} + \frac{1}{2^6\, 7} - \frac{1}{2^9\, 10} + \cdots$$

since the series is an alternating series and

$a_n = \dfrac{1}{2^{3n}(3n+1)} > 0$ is decreasing & $a_n \to 0$ as $n \to \infty$

We know that

$$|\text{Error}| = \left|\int_0^1 \frac{8}{8-x^3}\,dx - \left(1-\frac{1}{32}+\frac{1}{448}\right)\right| < \frac{1}{2^9\cdot 10} = \frac{1}{5120}$$

Name, Last name:

6. (18 pts) Consider the line $L: \dfrac{x-1}{-2} = \dfrac{y}{1} = \dfrac{z+2}{0}$, the plane $\mathcal{P}: x+2y-z=3$ and the point $A=(2,1,1)$.

(a) Does L intersect $\mathcal{P}$?

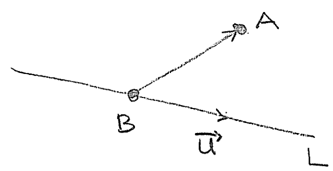
$x = 1-2t,\ y = t,\ z = -2$ should satisfy $x+2y-z=3$.

Since

$1-\cancel{2t}+\cancel{2t}+2 = 3$ for all $t \in \mathbb{R}$,

$L \in \mathcal{P}$.

(b) Find an equation of the plane $\mathcal{P}_1$ containing L and A .



Note that $A \notin L$.

Let $B = (1,0,-2) \in L$

$\overrightarrow{BA} = \langle 1, 1, 3 \rangle$

$\vec{u} = \langle -2, 1, 0 \rangle$

$$\vec{n}_1 = \overrightarrow{BA} \times \vec{u} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 3 \\ -2 & 1 & 0 \end{vmatrix} = \langle -3, -6, 3 \rangle$$

$$\mathcal{P}_1 : (-3)(x-1) - 6(y-0) + 3(z+2) = 0 \Rightarrow \underset{\mathcal{P}}{\overset{\mathcal{P}_1}{\shortparallel}} : x+2y-z=3$$

(c) Find an equation of the plane $\mathcal{P}_2$ containing L and perpendicular to $\mathcal{P}$

$$\vec{n}_2 = \vec{u} \times \vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 1 & 0 \\ 1 & 2 & -1 \end{vmatrix} = \langle -1, -2, -5 \rangle$$

$$\mathcal{P}_2 : (-1)(x-1) - 2(y-0) - 5(z+2) = 0$$

$$\Rightarrow \mathcal{P}_2 : x+2y+5z = -9$$

7. (3+6+6 pts) Let $f(x,y) = \begin{cases} \dfrac{x^3+2xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

(a) Explain why the function f is continuous at any $(a,b) \neq (0,0)$.

$\dfrac{x^3+2xy}{x^2+y^2}$ is a rational fcn. Since $x^2+y^2 \neq 0$ at $(a,b) \neq (0,0)$,

$f(x,y) = \dfrac{x^3+2xy}{x^2+y^2}$ is cont. at any $(a,b) \neq (0,0)$.

(b) Is the function f continuous at $(0,0)$?. Explain.

$$\lim_{(x,y)\to(0,0)} \frac{x^3+2xy}{x^2+y^2} = \lim_{\substack{(x,y)\to(0,0) \\ y=mx}} \frac{x^3+2mx^2}{x^2+m^2x^2}$$

$$= \lim_{\substack{(x,y)\to(0,0) \\ y=mx}} \frac{x+2m}{1+m^2} = \frac{2m}{1+m^2}$$

$m=0$, $m=1$ give different limits. Therefore,

$\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist

and hence

$f(x,y)$ is not cont. at $(0,0)$.

(c) Find the function $f_x(x,y)$. (Hint: Consider the cases $(x,y) \neq (0,0)$ and $(x,y) = (0,0)$ separately).

If $(x,y) \neq (0,0)$, $f(x,y) = \dfrac{x^3+2xy}{x^2+y^2}$. Then

$$f_x(x,y) = \frac{(3x^2+2y)\cdot(x^2+y^2) - 2x\,(x^3+2xy)}{(x^2+y^2)^2}$$

for $(x,y) \neq (0,0)$.

If $(x,y) = (0,0)$, we look at

$$\lim_{h\to 0} \frac{f(0+h,0) - \overset{0}{f(0,0)}}{h} = \lim_{h\to 0} \frac{f(h,0)}{h} = \lim_{h\to 0} \frac{\frac{h^3+0}{h^2+0}}{h}$$

$$= \lim_{h\to 0} 1$$

$$= 1$$

$\therefore$ $f_x(0,0) = 1$.

M E T U Department of Mathematics

Math 120 Calculus II Exam 2 30.07.2009		
Last Name: Name : Student No:	Section : Time : 19: 15 Duration : 90 *minutes*	Signature
5 QUESTIONS ON 5 PAGES		TOTAL 100 POINTS

1	2	3	4	5			

***(15+15 pts)* 1.** Let L_1 be the line with symmetric equations $x = \frac{y+11}{-7} = \frac{z+6}{-3}$ and let L_2 be the line with parametric equations $x = 1+s$, $y = 2s$, $z = 3+3s$.

(a) Show that L_1 and L_2 are not skew lines.

Substitute equations for L_2 in equation for L_1:

$1+s = \frac{2s+11}{-7}$ $\qquad$ $\frac{2s+11}{-7} = \frac{3+3s+6}{-3}$

$-7-7s = 2s+11$ $\qquad$ $-6s-33 = -63-21s$

$-18 = 9s$ $\qquad$ $15s = -30$

$s = -2$ $\qquad$ $s = -2$

since the solutions are the same, L_1 and L_2 have an intersection point. So they are not skew.

(b) Find an equation of the plane which contains both L_1 and L_2.

point: $s = -2$ $\quad (-1, -4, -3)$

$\vec{v_1} = \langle 1, -7, -3 \rangle$, $\vec{v_2} = \langle 1, 2, 3 \rangle$

$$\vec{n} = \vec{v_1} \times \vec{v_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -7 & -3 \\ 1 & 2 & 3 \end{vmatrix} = \langle -15, -6, 9 \rangle$$

equation:

$-15(x+1) - 6(y+4) + 9(z+3) = 0$

$-15x - 6y + 9z = 12$

$-5x - 2y + 3z = 4$

$5x + 2y - 3z = -4$

(13 pts) **2.** Find the length of the curve $\vec{r}(t) = < e^t \cos t, e^t \sin t, e^t >$, $0 \le t \le \ln 2$.

$$\vec{r}\,'(t) = \langle e^t\cos t - e^t\sin t,\ e^t\sin t + e^t\cos t,\ e^t \rangle$$

$$|\vec{r}\,'(t)| = \sqrt{e^{2t}\cos^2 t - 2e^{2t}\cos t\sin t + e^{2t}\sin^2 t + e^{2t}\sin^2 t + 2e^{2t}\sin t\cos t + e^{2t}\cos^2 t + e^{2t}}$$

$$= \sqrt{3e^{2t}} = \sqrt{3}\, e^t$$

$$L = \int_0^{\ln 2} \sqrt{3}e^t\,dt = \sqrt{3}e^t \Big|_0^{\ln 2} = \sqrt{3}\,(2-1) = \sqrt{3}$$

(12 pts) **3.** Let $f(x,y) = x^2 + y^2 - 2x - 4y$.

(a) Find the gradient vector of f at the point $(-2,3)$.

$\vec{\nabla} f(x,y) = \langle 2x-2,\ 2y-4 \rangle$

$\vec{\nabla} f(-2,3) = \langle -6, 2 \rangle$

(b) Find the directional derivative of f at the point $(-2,3)$ in the direction of the vector $\vec{v} = 3\vec{\imath} - 4\vec{\jmath}$.

$|\vec{v}| = 5$

$\vec{u} = \frac{1}{|\vec{v}|}\vec{v} = \langle \frac{3}{5}, -\frac{4}{5} \rangle$

$D_{\vec{u}} f(-2,3) = \vec{\nabla} f(-2,3) \cdot \vec{u}$

$= \langle -6, 2 \rangle \cdot \langle \frac{3}{5}, -\frac{4}{5} \rangle$

$= -\frac{18}{5} - \frac{8}{5} = -\frac{26}{5}$

(15 pts) **4.** Find an equation for the tangent plane to the graph of $z = f(xy, x+y, x^2+y^2)$ at $(x,y) = (1,2)$ if $f(2,3,5) = 1$, $f_1(2,3,5) = -4$, $f_2(2,3,5) = 7$, $f_3(2,3,5) = 10$.

$$\frac{\partial z}{\partial x} = f_1 \cdot y + f_2 \cdot 1 + f_3 \cdot 2x$$

$$\frac{\partial z}{\partial x}(1,2) = f_1(2,3,5) \cdot 2 + f_2(2,3,5) + 2 f_3(2,3,5)$$

$$= -8 + 7 + 20 = 19$$

$$\frac{\partial z}{\partial y} = f_1 \cdot x + f_2 \cdot 1 + f_3 \cdot 2y$$

$$\frac{\partial z}{\partial y}(1,2) = (-4) + 7 + 10 \cdot 4 = 43$$

$$z(1,2) = f(2,3,5) = 1$$

$$z = 1 + 19(x-1) + 43(y-2)$$

$$z = 19x + 43y - 104$$

(15+15 pts) 5 Let $f(x,y) = \begin{cases} \frac{5x^3+xy^4}{2\sqrt{x^4+y^4}} & , \quad (x,y) \neq (0,0) \\ 0 & , \quad (x,y) = (0,0) \end{cases}$.

(a) Show that the function f is continuous at the point $(0,0)$.

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{5x^3+xy^4}{2\sqrt{x^4+y^4}}$$

$$\left|\frac{5x^3+xy^4}{2\sqrt{x^4+y^4}}\right| = \frac{5|x|x^2+|x|y^4}{2\sqrt{x^4+y^4}}.$$

Since $x^4 \le x^4+y^4$, $x^2 \le \sqrt{x^4+y^4}$ and $\frac{x^2}{\sqrt{x^4+y^4}} \le 1$.

similarly $\frac{y^2}{\sqrt{x^4+y^4}} \le 1$.

so

$$0 \le \left|\frac{5x^3+xy^4}{2\sqrt{x^4+y^4}}\right| \le \frac{5|x|}{2} + \frac{|x|}{2}$$

$\lim_{(x,y)\to(0,0)} 0 = 0$

$\lim_{(x,y)\to(0,0)} \left(\frac{5|x|}{2} + \frac{|x|}{2}\right) = 0$

By sequeeze Theorem

$\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$

so f is continuous at $(0,0)$.

(b) Find $f_x(0,0)$ and $f_y(0,0)$ if they exist.

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{5h^3}{2\sqrt{h^4}} - 0}{h}$$

$$= \lim_{h\to 0} \frac{5}{2} = \frac{5}{2}$$

$(x,y) \neq (0,0)$ $\quad f_x(x,y) = \frac{\partial}{\partial x}\left(\frac{5x^3+xy^4}{2\sqrt{x^4+y^4}}\right)$

$$= \frac{(15x^2+y^4)\,2\sqrt{x^4+y^4} - (5x^3+xy^4)\frac{1}{\sqrt{x^4+y^4}}\cdot 4x^3}{4(x^4+y^4)}$$

$$= \frac{2(15x^2+y^4)(x^4+y^4) - 4x^3(5x^3+xy^4)}{4(x^4+y^4)^{3/2}}$$

$$f_{xy}(0,0) = \lim_{h\to 0} \frac{f_x(0,h)-f_x(0,0)}{h} = \lim_{h\to 0} \frac{\frac{\cancel{2}\,\cancel{h}^8}{2\cancel{4}\,(h^4)^{\cancel{3}/2}} - \frac{5}{2}}{h} = \lim \frac{-2}{1}\ \text{d.n.}$$

1956

Math 120- Calculus for Functions of Several Variables
Final Exam (120 minutes)

Name :
Lastname :

1	2	3	4	5	6	7	T

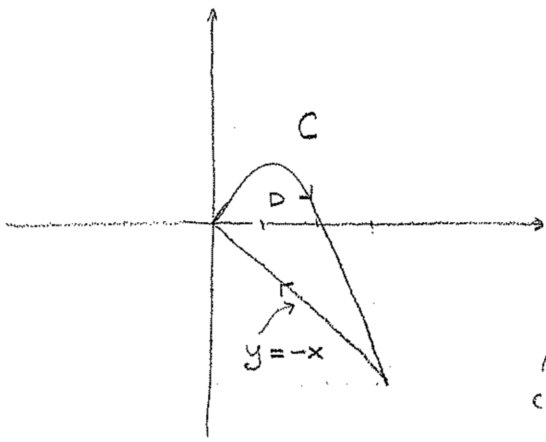
NO PARTIAL CREDIT WILL BE GIVEN FOR UNSUPPORTED ANSWERS!

---

1. (15 pts) Use Green's theorem to evaluate

$$\oint_C (x \sin x - x^2 y)\, dx + (2xy - x^2 + \cos y)\, dy,$$

where C consists of the arc of the curve $y = 2x - x^2$ from (0,0) to (3,-3) and the line segment from (3,-3) to (0,0). Explain your answer.



$P = x\sin x - x^2 y$

$Q = 2xy - x^2 + \cos y$

$Q_x - P_y = 2y - 2x + x^2$

$-C$ is positively oriented, simple, closed piecewise smooth and P and Q have cont first order partial derivatives everywhere. Thus, we may apply Green's thm.

$$I = -\iint_D (2y - 2x + x^2)\, dA$$

$$= -\int_0^3 \left[\int_{-x}^{2x-x^2} (2y - 2x + x^2)\, dy\right] dx$$

$$= -\int_0^3 \left[y^2 - (2x - x^2) y \right]\Big|_{-x}^{2x-x^2} dx$$

$$= \int_0^3 \left[x^2 + (2x - x^2)x\right] dx$$

$$= x^3 - \frac{x^4}{4} \Big|_0^3$$

$$= 27 - \frac{81}{4} = \frac{27}{4}$$

2. (10 pts)

(a) Show that the planes $x+y-z=1$ and $2x-3y+4z=5$ are neither parallel nor perpendicular.

$\vec{n}_1 = \langle 1, 1, -1 \rangle$, $\vec{n}_2 = \langle 2, -3, 4 \rangle$

$\vec{n}_1 \neq k\,\vec{n}_2$ for any $k \neq 0$ $\Rightarrow$ the planes are not parallel

$\vec{n}_1 \cdot \vec{n}_2 = 2-3-4 = -5 \neq 0$ $\Rightarrow$ the planes are not perpendicular.

(b) Find the line of intersection of the given planes.

Let $x = t$. Then

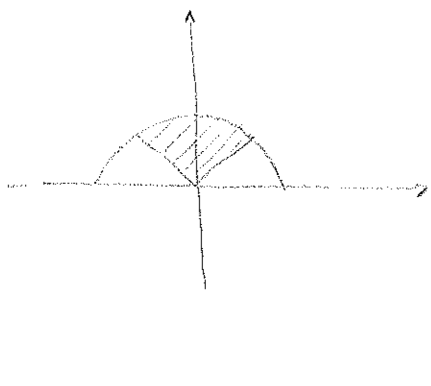
$$y - z = 1 - t$$
$$-3y + 4z = 5 - 2t$$

$$y = 4(1-t) + 5 - 2t$$
$$= 9 - 6t$$

$$z = 9 - 6t - (1-t)$$
$$z = 8 - 5t$$

ℓ: $x = t$, $y = 9 - 6t$, $z = 8 - 5t$

3. (10 pts) Evaluate the integral $\iint_D (x+y)\,dA$, where D is the region bounded by $y=\sqrt{4-x^2}$, $y=-x$, and $y=x$.



$$I = \iint_D (x+y)\,dA = \int_{r=0}^{2} \int_{\theta=\pi/4}^{3\pi/4} (r\cos\theta + r\sin\theta)\, r\,dr\,d\theta$$

$x = r\cos\theta$

$y = r\sin\theta$

$dA = r\,dr\,d\theta$

$$I = \int_0^2 r^2 (\sin\theta - \cos\theta)\Big|_{\pi/4}^{3\pi/4} dr$$
$$= \frac{8}{3}(\sqrt{2} - 0)$$
$$= \frac{8\sqrt{2}}{3}$$

4. (15 pts) Let $f(x,y) = \dfrac{3xy^2}{x^2+2y^4}$ when $(x,y) \neq (0,0)$, and $f(0,0) = 0$.

(a) At which points is f continuous? Since f is a rational function, it is continuous when $x^2+2y^4 \neq 0 \to (x,y) \neq (0,0)$. Thus, we need to check $(0,0)$. So we need to check the continuity at $(0,0)$.

$$\lim_{\substack{(x,y)\to(0,0) \\ (x=my^2)}} \frac{3xy^2}{x^2+2y^4} = \lim_{(x,y)\to(0,0)} \frac{3m\cancel{y^4}}{(m^2+2)\cancel{y^4}} = \frac{3m}{m^2+2}$$

Choosing m, we see that the limit does not exist. Hence, f is not cont. at $(0,0)$.

$\therefore$ f is cont. everywhere except at $(0,0)$.

(b) Find an equation of the tangent plane to the surface $z = f(x,y)$ at $(2,-1)$.

$$f_x = \left.\frac{3y^2(x^2+2y^4) - 2x(3xy^2)}{(x^2+2y^4)^2}\right|_{(2,-1)} = \frac{-6}{36} = -\frac{1}{6}$$

$$f_y = \left.\frac{6xy(x^2+2y^4) - 8y^3(3xy^2)}{(x^2+2y^4)^2}\right|_{(2,-1)} = \frac{-24}{36} = -\frac{2}{3}$$

$$\vec{n} = \langle -\tfrac{1}{6}, -\tfrac{2}{3}, -1 \rangle$$

Tangent plane : $-\frac{1}{6}(x-2) - \frac{2}{3}(y+1) - (z-1) = 0$

(c) Estimate the value $f(2.01, -1.05)$ by making use of (b).

$$z = -\frac{1}{6}x - \frac{2}{3}y + \frac{2}{3}$$

$$f(2.01, -1.05) \approx -\frac{1}{6}(2.01) - \frac{2}{3}(-1.05) + \frac{2}{3}$$ ✓

5. (15 pts) Let $f(x,y) = x^3 + 3y^2x - 12xy$

(a) Find and classify the critical points of f.

$f_x = 3x^2 + 3y^2 - 12y = 0 \quad (1)$

$f_y = 6yx - 12x = 0 \quad (2)$

$f_{xx} = 6x, \quad f_{yy} = 6x, \quad f_{xy} = f_{yx} = 6y - 12$

$(2) \Rightarrow x = 0$ or $y = 2$

Let $x = 0$. From (1) we have $3y^2 - 12y = 0$. So, $y = 0, y = 4$

If $y = 2$, then (1) implies that $3x^2 - 12 = 0$, and hence $x = \mp 2$

$\therefore$ The critical pts are $(0,0), (0,4), (-2,2)$, and $(2,2)$

$\Delta = f_{xx} f_{yy} - f_{xy}^2 = 36x^2 - (6y-12)^2$

$\Delta|_{(0,0)} < 0, \quad \Delta|_{(0,4)} < 0, \quad \Delta|_{(-2,2)} > 0, \quad \Delta|_{(2,2)} > 0$

$f_{xx}(-2,2) < 0, \quad f_{xx}(2,2) > 0$

By the SDT,

$f(-2,2) = 16$ (local max), $f(2,2) = 8$ (local min),

$(0,0)$ and $(0,4)$ are saddle pts.

(b) Does there exist absolute extrema of f in the xy-plane? Does this contradict the extreme value theorem? Why?

Let $y = 0$. Then $f(x,0) = x^3 - 2 \longrightarrow \mp\infty$ as $x \longrightarrow \mp\infty$

The fcn $f(x,y)$ is unbounded from above and below, which implies that absolute extrema do not exist.

No contradiction. Although f is cont. everywhere, its domain is not bounded!

6. (5+10+5=20 pts)

(a) Show that $\vec{F}(x,y) = (4x^3y^2 - 2xy^3 + \sin\pi x)\vec{i} + (2x^4y - 3x^2y^2 + 7y^6)\vec{j}$ is conservative in the xy-plane.

$P = 4x^3y^2 - 2xy^3 + \sin\pi x$, $Q = 2x^4y - 3x^2y^2 + 7y^6$. Since $P_y = 8x^3y - 6xy^2 = Q_x$ & all fcns are cont in the xy-plane which is open and simply connected, the vector field $\vec{F}$ is conservative.

(b) Find a potential ϕ for which $\vec{F} = \vec{\nabla}\phi$, and evaluate the line integral

$$I = \int_C (4x^3y^2 - 2xy^3 + \sin\pi x)\, dx + (2x^4y - 3x^2y^2 + 7y^6)\, dy$$

from (0,0) to (1,1) along the curve $C_1 : \ y = \sin(\pi x/2)$.

$\phi_x = P = 4x^3y^2 - 2xy^3 + \sin\pi x \quad (1)$

$\phi_y = Q = 2x^4y - 3x^2y^2 + 7y^6 \quad (2)$

$(1) \Rightarrow \phi = x^4y^2 - x^2y^3 - \frac{1}{\pi}\cos\pi x + h(y)$. Then from (2) &

$\phi_y = 2x^4y - 3x^2y^2 + h'(y)$, we have $h'(y) = 7y^6$.

We may take $h(y) = y^7$. Thus, $\phi = x^4y^2 - x^2y^3 - \frac{1}{\pi}\cos\pi x + y^7$.

By the Fundamental thm of line integrals,

$I = \phi\big|_{(1,1)} - \phi\big|_{(0,0)} = \frac{1}{\pi} + 1 - \left(-\frac{1}{\pi}\right) = \frac{2}{\pi} + 1$

(c) Find the line integral

$$J = \int_C (2y + 4x^3y^2 - 2xy^3 + \sin\pi x)\, dx + (2x^4y - 3x^2y^2 + 7y^6)\, dy$$

from (0,0) to (1,1) along the curve $C_2 : \ y = x^3$.

$$J = \int_{C_2} 2y\, dx + \int_{C_2} \vec{F}\cdot d\vec{r} = \int_{C_2} 2y\, dx + \int_{C_1} \vec{F}\cdot d\vec{r}$$

$$= \int_{C_2} 2y\, dx + \left(\frac{2}{\pi} + 1\right)$$

$C_2 : \ x = t,\ y = t^3,\ t \in [0,1]$

$$J = \int_0^1 2t^3\, dt + \left(\frac{2}{\pi} + 1\right) = \frac{3}{2} + \frac{2}{\pi}$$

7. (15 pts) Test for convergence:

(a) $a_n = \dfrac{n!}{1\cdot 3\cdot\cdot 5\cdots(2n+1)}$

$$0 \leq a_n = \frac{1\cdot 2\cdot 3\cdot\cdots\cdot n}{1\cdot 3\cdot 5\cdot\cdots 2n-1}\,\frac{1}{2n+1} \leq \frac{1}{2n+1}$$

$$\lim_{n\to\infty}\frac{1}{2n+1} = 0, \quad \lim_{n\to 0} 0 = 0. \quad \text{By the ST,} \quad \lim_{n\to\infty} a_n = 0$$

(b) $\displaystyle\sum_{n=1}^{\infty} b_n$, where $\displaystyle b_n = \sum_{k=0}^{n}\frac{1}{k!}$.

$$\lim_{n\to\infty} b_n = \sum_{k=0}^{\infty}\frac{1}{k!} = \sum_{k=0}^{\infty}\frac{1^k}{k!} = e^1 \neq 0$$

By the n-th term test, the series is D.

(c) $\displaystyle\sum_{n=1}^{\infty}[\tan^{-1}(n+1) - \tan^{-1}(n)]$.

$$S_n = \sum_{k=1}^{n} \tan^{-1}(k+1) - \tan^{-1}k = \tan^{-1}(n+1) - \tan^{-1}1$$

$$= \tan^{-1}(n+1) - \frac{\pi}{4}$$

$$\lim_{n\to\infty} S_n = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} \quad \Rightarrow \quad \text{The series is C.}$$

<u>OR</u>

Compare with $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$