M E T U
Department of Mathematics

Group	CALCULUS II MidTerm 1	List No.

Code : *Math 120*
Acad. Year : *2009-2010*
Semester : *Spring*
Coordinator : *Muhiddin Uğuz*

Date : *April.10.2010*
Time : *09:30*
Duration : *150 minutes*

Last Name :
Name : Student No. :
Department : Section :
Signature :

9 QUESTIONS ON 6 PAGE
TOTAL 100(+5 Bonus) POINTS

1	2	3	4	5	6	7	8	9	

Question 1 (15 points) Determine whether the following sequences are convergent or not. Give your explanations.

a) $\left\{\frac{2^n(n!)^2}{(2n)!}\right\}_{n=0}^{\infty} = a_n$. Consider $\sum a_n$. If this series is convergent, then by n^{th} term test, $\lim_{n\to\infty} a_n = 0$, that is sequence a_n is convergent.

Let's apply ratio test to $\sum a_n$. (since $a_n > 0$ $\forall n$, we can use ratio test)

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} 2^{\cancel{n+1}} \frac{(n+1)!^2}{(2n+2)!} \cdot \frac{(2n)!}{\cancel{2^n}\cdot(n!)^2} = \lim_{n\to\infty} \frac{2(n+1)^2 \cdot \cancel{(n!)^2}}{(2n+2)(2n+1)\cancel{(2n)!}} \frac{\cancel{(2n)!}}{\cancel{(n!)^2}}$$

$= \frac{1}{2} < 1$ Thus $\sum a_n$ is convergent series

$\therefore \{a_n\}$ is convergent sequence $(\lim_{n\to\infty} a_n = 0)$

OR/ $$0 \le a_n = \frac{\cancel{1.2\cdots n}}{\cancel{1.2\cdots n}} \frac{2.1}{(n+1)} \overset{\le 1}{\frac{2.2}{(n+2)}} \cdots \overset{\le 1}{\frac{2(n-2)}{(2n-2)}} \overset{\le 1}{\frac{2.(n-1)}{(2n-1)}} \overset{=1}{\frac{2.n}{(2n)}} \le \frac{2}{n+1} \to 0 \Rightarrow$$ by squeezing thm, $\lim_{n\to\infty} a_n = 0$

b) $a_1 = 120$, $a_{n+1} = a_n + \sqrt{a_n}$ for $n = 1, 2, 3, \ldots$

First note that $a_{n+1} - a_n = \sqrt{a_n} \geq 0$. This means a_n is an increasing sequence and $a_1 = 120$. Therefore $a_n \geq 120$ $\forall n$.

If $\boxed{\lim_{n\to\infty} a_n \text{ exists,}}$ say, is L, then $\boxed{L \geq 120}$

On the other hand $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n \Rightarrow \cancel{L} = \cancel{L} + \sqrt{L} \Rightarrow \boxed{L = 0 < 120}$

a contradiction.

$\therefore \lim_{n\to\infty} a_n$ does not exist.

c) $\left\{(\ln n)^{\frac{1}{n}}\right\}_{n=2010}^{\infty}$

Let $f(x) = (\ln x)^{1/x}$. Then $f(n) = a_n$ $\forall n = 2010, \ldots$

(If $\lim_{x\to\infty} f(x) = L$ (L is possibly ∞), then $\lim_{n\to\infty} a_n = L$.)

$$\lim_{x\to\infty} (\ln x)^{1/x} = e^{\lim_{x\to\infty} \frac{1}{x}\ln(\ln x)} = e^{\lim_{x\to\infty} \frac{\ln(\ln x)}{x}} \quad \left(\frac{\infty}{\infty} \text{ type}\right)$$

$$= e^{\lim_{x\to\infty} \frac{1}{x \ln x}} = e^0 = 1 \Rightarrow \lim_{n\to\infty} a_n = 1$$

↑ L'Hop's Rule

so a_n is convergent sequence

Question 2 (15 points) Determine whether the following series are convergent or not. Give your explanations.

a) $\sum_{n=219}^{\infty}\left[\sin(\frac{1}{n^2})+\frac{1}{n(\ln n)^3}\right]$

Consider $\sum \frac{1}{n^2}$ and $\sum \frac{1}{n(\ln n)^3}$. If each one is convergent (say first one is A and second one is B, then given series is also convergent and equal to A+B.

- $\sum \underset{a_n}{\boxed{\sin\frac{1}{n^2}}}$. Note that $\frac{1}{n^2}>0$ and $\lim_{n\to\infty}\frac{1}{n^2}=0 \Rightarrow a_n=\sin\frac{1}{n^2}>0$

so, we can use limit comparison test: $\lim_{n\to\infty}\frac{\sin\frac{1}{n^2}}{\frac{1}{n^2}}=\lim_{x\to 0}\frac{\sin x}{x}=1$

since limit is a nonzero finite number, either both $\sum\sin\frac{1}{n^2}$ and $\sum\frac{1}{n^2}$ converge or both diverge. Since $\sum\frac{1}{n^2}$ is convergent (by p-test) so is $\sum\sin\frac{1}{n^2}$.

- $\sum \underset{b_n}{\boxed{\frac{1}{n.\ln^3 n}}}$. Let $f(x)=\frac{1}{x\ln^3 x}$. Then

since $\begin{cases} f(n)=b_n\ \forall n, & f(x) \text{ is continuous on } [219,\infty) \\ f'(x)<0, \text{ hence} & f(x) \text{ is decreasing on } [219,\infty) \\ \lim_{x\to\infty} f(x)=0 \end{cases}$

we can say (by integral test, either both $\sum b_n$ and $\int_{219}^{\infty} f(x)dx$ converge or both diverge.

$$\int_{219}^{\infty}\frac{1}{x\ln^3 x}dx \quad \lim_{c\to\infty}\int_{219}^{c}\frac{1}{x\ln^3 x}dx=\lim_{c\to\infty}\int_{\ln 219}^{\ln c}\frac{1}{u^3}du \quad \text{conv. by p-test } (p=3>1)$$

$u=\ln x$
$du=\frac{1}{x}dx$

$\therefore$ being sum of two convergent series, $\sum a_n+b_n$ is also convergent

b) $\sum_{n=0}^{\infty}(-1)^n \underset{a_n}{\boxed{\frac{2n+3}{(n+1)(n+2)}}}$

$\sum(-1)^n a_n$ is an alternating series $(a_n>0)$

- $\lim_{n\to\infty} a_n=\lim_{x\to\infty}\overset{f(x)}{\boxed{\frac{2x+3}{(x+1)(x+2)}}}=0$
- $f(x)=\frac{2x+3}{x^2+3x+2} \Rightarrow f'(x)=\frac{2(x^2+3x+2)-(2x+3)(2x+3)}{(x^2+3x+2)^2}=\frac{2x^2+6x+4-4x^2-12x-9}{(x^2+3x+2)^2}$

$=\frac{-2x^2-6x-5}{(--)^2}<0 \Rightarrow f(x)\searrow$

$\Rightarrow$ a_n is decreasing

Hence by Alternating Series Test, $\sum(-1)^n a_n$ is convergent.

(note that $\sum|(-1)^n a_n|=\sum\frac{2n+1}{(n+1)(n+2)}$ is divergent (by limit comparison test with $\frac{1}{n}$)

Hence $\sum(-1)^n a_n$ is in fact conditionally convergent)

c) $\sum_{n=1}^{\infty}\underset{a_n}{\boxed{\sin(1-e^{-n})}}$

$$\lim_{n\to\infty} a_n=\lim_{n\to\infty}\sin(1-e^{-n})\overset{\sin x \text{ is continuous}}{=}\sin(1-\lim_{n\to\infty}e^{-n})=\sin 1\neq 0$$

Hence by n^{th} term test, $\sum a_n$ is divergent

Question 3 (10 points) Determine whether the followings satatements are true or false. Give explanations.

a) If $\sum_{n=1}^{\infty} n^2 a_n$ is convergent and $a_n > 0 \ \forall n$, then $\sum_{n=1}^{\infty} a_n$ is also convergent.

First, since $\sum n^2 a_n$ is convergent, by n^{th} term test, $\lim_{n\to\infty} n^2 a_n = 0$.

Since $\lim_{n\to\infty} \frac{a_n}{\frac{1}{n^2}} = \lim_{n\to\infty} n^2 a_n = 0$, and $\sum \frac{1}{n^2}$ is convergent (by p-test),

by Limit Comparison test, $\sum a_n$ is also convergent.

b) If $\sum_{n=1}^{\infty} a_n$ converges then $\sum_{n=1}^{\infty} \left(\frac{1}{7^{a_n}}\right)$ is also convergent.

Since $\sum a_n$ converges, $\lim_{n\to\infty} a_n = 0 \Rightarrow \lim_{n\to\infty} 7^{a_n} = 1$

$\Rightarrow \lim_{n\to\infty} \frac{1}{7^n} = 1 \neq 0 \Rightarrow \sum \frac{1}{7}$ diverges

Question 4 (10 points)

Determine the set of points at which $f(x,y) = \begin{cases} \frac{x^2}{2x^2+y^2} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases}$ is continuous.

- If $(x,y) \neq (0,0)$, f is given by ratio of two polynomials and denominator is <u>not</u> zero. Since polynomials are continuous everywhere f is also continuous.
- if $(x,y) = (0,0)$; (f is continuous if ① $f(0,0)$ is defined ② $\lim_{(x,y)\to(0,0)} f(x,y) = L$ exists ③ $L = f(0,0)$)

$\lim_{\substack{(x,y)\to(0,0) \\ \text{along } y=mx}} f(x,y) = \lim_{x\to 0} f(x,mx) = \lim_{x\to 0} \frac{x^2}{2x^2+m^2x^2} = \frac{1}{2+m^2}$ depends on m

so limit does not exist

Hence f is <u>not</u> continuous at $(0,0)$

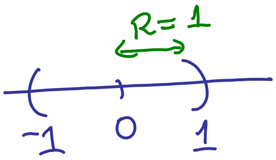
f is continuous on $\mathbb{R}^2 \setminus \{(0,0)\}$

Question 5 (10 points) Find the radius **R** of convergence and the interval **I** of convergence of the power series $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{\sqrt{n+1}}$.

This is a power series about $a=0$. To find it's radius of convergence we apply ratio test;

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|x|^{\cancel{n+1}}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{\cancel{|x|^n}} = |x| \lim_{n\to\infty} \sqrt{\frac{n+1}{n+2}} = |x|$$

given power series is convergent if $|x| < 1$



Thus $R=1$

To find interval of convergence, we check the end points $x = \mp 1$

- $x=1 \Rightarrow \sum \frac{(-1)^n}{\sqrt{n+1}}$ is convergent by Alternating series test $\left(\lim_{n\to\infty} \frac{1}{\sqrt{n+1}} = 0 \text{ and } \frac{1}{\sqrt{n+1}} \text{ is decreasing} \right)$

$f(x) = \frac{1}{\sqrt{x+1}} = (x+1)^{-1/2} \Rightarrow f'(x) = \frac{-1}{2}(x+1)^{-3/2} < 0$
$\Rightarrow f(x)$ decreasing

- $x=-1 \Rightarrow \sum \frac{1}{\sqrt{n+1}}$ is divergent (Limit comp. test with $\sum \frac{1}{n^{1/2}}$)

Thus $I = (-1, +1]$

Question 6 (10 points) Find the 120^{th} derivative $f^{(120)}(x)$ of $f(x) = \sin(x^3)$ at $x=0$.

$$\sin x^3 = \sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{6n+3}$$

at $x=0$

$f(x) = \sin x = f^{(4k)}(x) = 0$
$f'(x) = \cos x = f^{(4k+1)}(x) = 1$
$f^{(2)} = -\sin x = f^{(4k+2)}(x) = 0$
$f^{(3)} = -\cos x = f^{(4k+3)}(x) = -1$

$\Downarrow$

$$f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f^{(3)}(0)x^3}{3!} + \cdots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sin x \quad \forall x$$

note that we don't have terms x^k when k is an even number; in other words, coefficient of even power terms are all zero.

Also

$$\sin x^3 = f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \cdots + \frac{f^{(120)}(0)}{120!} x^{\boxed{120}} + \cdots$$

even number

$\Downarrow$

coefficient must be zero

$\Downarrow$

$f^{(120)}(0) = 0$

Question 7 (10 points) Find the exact value of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n2^n}$.

Consider $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n \quad \forall |x|<1$

$$\Rightarrow \int_0^x \frac{1}{1+t}\,dt = \sum_{n=0}^{\infty} (-1)^n \int_0^x t^n\,dt \Rightarrow \ln|1+x| = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} \quad \forall |x|<1$$

$$\Rightarrow \ln|1+x| = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} \quad \forall |x|<1$$

put $x = \frac{1}{2} \Rightarrow \ln\frac{3}{2} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n \cdot 2^n}$

Question 8 (10 points) How many nonzero terms in the Maclaurin series for e^{-x^4} are needed to evaluate $\int_0^{1/2} e^{-x^4}dx$ to guarantee that the error is less than $\frac{1}{2010}$.

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \forall x$

$\left(f(x) = e^x \Rightarrow f^{(k)}(x) = e^x \Rightarrow f^{(k)}(0) = 1 \quad \forall k = 0,1,2,\dots \Rightarrow \sum \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^{\infty} \frac{x^k}{k!} \right)$

$$\Rightarrow e^{-x^4} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{n!} \quad \forall x$$

$$\Rightarrow \int_0^{1/2} e^{-x^4}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{x^{4n+1}}{4n+1} \Bigg|_0^{1/2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(4n+1)\,2^{4n+1}}$$

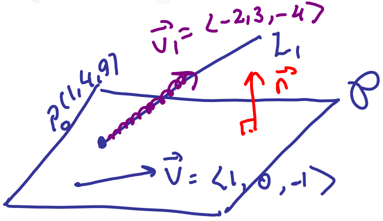
$$= \frac{1}{2} - \frac{1}{5 \cdot 2^5} + \overset{< \frac{1}{2010}}{\boxed{\frac{1}{2 \cdot 9 \cdot 2^9}}} - \dots \dots$$

$$\underset{\substack{\uparrow \\ \text{error} < \frac{1}{2\cdot 9\cdot 2^9} < \frac{1}{2010}}}{\approx} \frac{1}{2} - \frac{1}{5 \cdot 32}$$

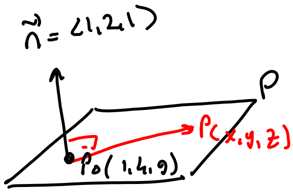
so 2 tems is enough

Question 9 (15 points) Given $\vec{v} = \vec{i} - \vec{k}$ and $L_1 :\ (1-2t)\vec{i} + (4+3t)\vec{j} + (9-4t)\vec{k}$.

a) Find an equation of the plane **P** which is parallel to the vector $\vec{v}$ and containing the line L_1.



$$\vec{n} \parallel \vec{v} \times \vec{v}_1 = \begin{vmatrix} i & j & k \\ 1 & 0 & -1 \\ -2 & 3 & -4 \end{vmatrix} = \langle 3, 6, 3 \rangle \Rightarrow \text{we can take } \vec{n} = \langle 1,2,1 \rangle$$



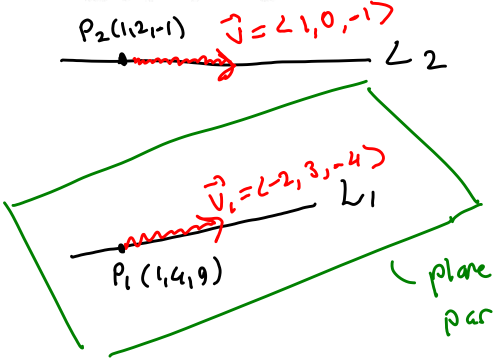
$$P(x,y,z) \in P \Leftrightarrow \vec{n} \cdot \vec{P_0P} = 0$$

$$\langle x-1, y-4, z-9 \rangle \cdot \langle 1,2,1 \rangle = 0$$

$$x-1+2y-8+z-9=0$$

$$x+2y+z=18$$

b) Find the distance between the line L_1 and the line L_2 which is passing through the point $P_2(1,2,-1)$ and parallel to the vector $\vec{v} = \vec{i} - \vec{k}$.



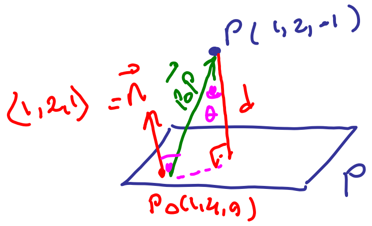
(plane containing L_1, parallel to L_2

$\Downarrow$ ← part (a)

$\underline{P}$: $x+2y+z=18$

any point on L_2

$$\text{Distance} = d(L_1, L_2) = d(P_2, \underline{P})$$



$$\vec{P_0P} = \langle 0,-2,-10 \rangle$$

$$\vec{n} = \langle 1,2,1 \rangle$$

$$|\cos\theta| = \frac{d}{|\vec{P_0P}|}$$

$$\Downarrow$$

$$d = |\vec{P_0P}| \frac{|\vec{n}|}{|\vec{n}|} |\cos\theta|$$

$$= \frac{|\vec{P_0P} \cdot \vec{n}|}{|\vec{n}|}$$

$$= \frac{|-4-10|}{\sqrt{6}} = \boxed{\frac{14}{\sqrt{6}}}$$

c) Given the curve with parametric equation $\vec{r}(t) = (t^2 - t^3, 1-3t, t^3)$; $t \in \Re$, find the point(s) of intersection of $\vec{r}(t)$ and $x + 2y + z = 18$.

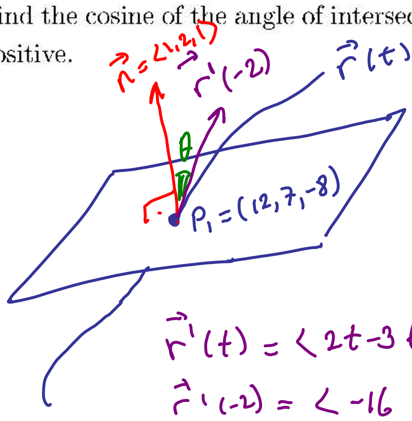
$$\vec{r}(t) = (x(t), y(t), z(t)) = (t^2 - t^3, 1-3t, t^3) \quad \underline{\underline{\text{and}}} \quad x+2y+z=18$$

$$\Rightarrow t^2 - \cancel{t^3} + 2(1-3t) + \cancel{t^3} = 18 \Rightarrow t^2 - 6t - 16 = 0 = (t-8)(t+2)$$

$$\Rightarrow t_1 = 8, \quad t_2 = -2$$

$$\Rightarrow P_1 = (-448, -23, 512) \text{ and } P_2(12, 7, -8)$$

d) Find the cosine of the angle of intersection at the point in part (c) where the x-coordinate is positive.



$$\vec{n} \cdot \vec{r}'(-2) = \|\vec{n}\| \, \|\vec{r}'(-2)\| \cos\theta$$

$$\cos\theta = \frac{\vec{n} \cdot \vec{r}'(-2)}{\|\vec{n}\| \, \|\vec{r}'(-2)\|} = \frac{-16-6+12}{\sqrt{6} \cdot \sqrt{309}} = \frac{-10}{\sqrt{6 \cdot 309}}.$$

$$\vec{r}'(t) = \langle 2t - 3t^2, -3, 3t^2 \rangle$$

$$\vec{r}'(-2) = \langle -16, -3, 12 \rangle$$

METU
Department of Mathematics

Group	CALCULUS II MidTerm 2	List No.

Code : *Math 120*
Acad. Year : *2009-2010*
Semester : *Spring*
Coordinator: *Muhiddin Uğuz*

Date : *May.22.2010*
Time : *09:30*
Duration : *120 minutes*

Last Name :
Name : Student No. :
Department : Section :
Signature :

5 QUESTIONS ON 4 PAGE
TOTAL 100 POINTS

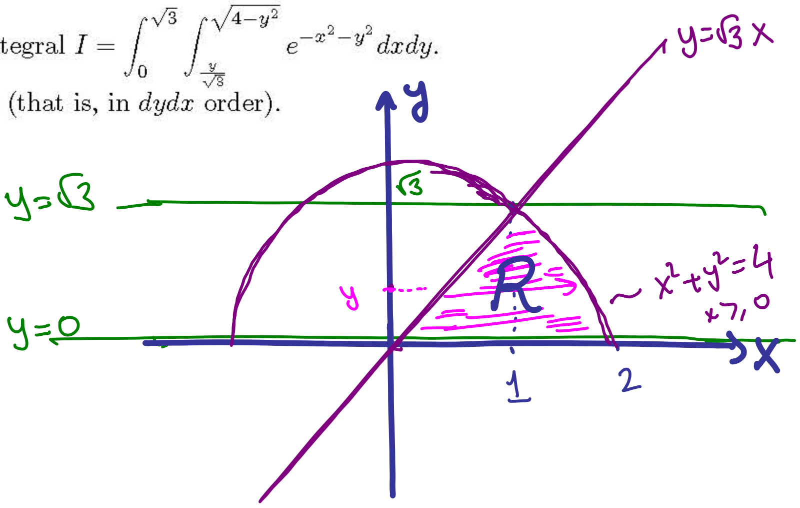
1	2	3	4	5		

Question 1 (24 points) Consider the integral $I = \int_0^{\sqrt{3}} \int_{\frac{y}{\sqrt{3}}}^{\sqrt{4-y^2}} e^{-x^2-y^2}\,dx\,dy$.

a) Write I as a double integral in reverse order (that is, in $dydx$ order).

$$I = \int_0^1 \int_0^{\sqrt{3}x} e^{-x^2-y^2}\,dy\,dx +$$

$$+ \int_1^2 \int_0^{\sqrt{4-x^2}} e^{-x^2-y^2}\,dy\,dx$$

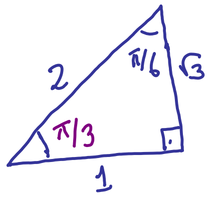


$x = \frac{y}{\sqrt{3}} \Leftrightarrow y = \sqrt{3}x$

$x = \sqrt{4-y^2} \Leftrightarrow (x^2+y^2=4 \text{ and } x \geq 0)$

b) Write I as a double integral in polar coordinates.

$$I = \int_0^{\pi/3} \int_0^2 e^{-r^2} \cdot r\,dr\,d\theta$$



c) Evaluate I.

$$I = \int_0^{\pi/3} 1\,d\theta \cdot \int_0^2 e^{-r^2} r\,dr = \frac{\pi}{3}\left(\frac{-1}{2}\int_0^{-4} e^{u}\,du\right)$$

$u = -r^2$

$du = -2\boxed{r\,dr}$

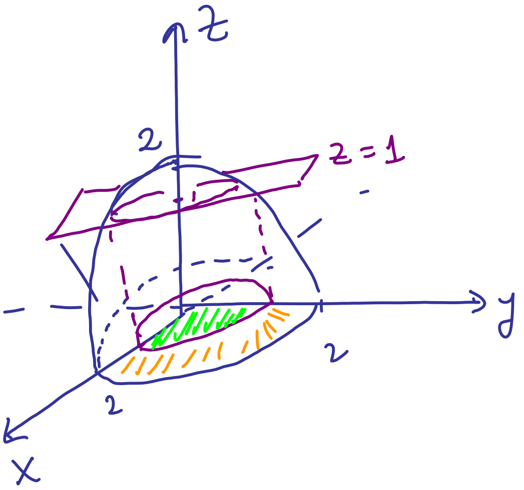
$$= \frac{\pi}{6} e^{u}\Big|_{-4}^{0}$$

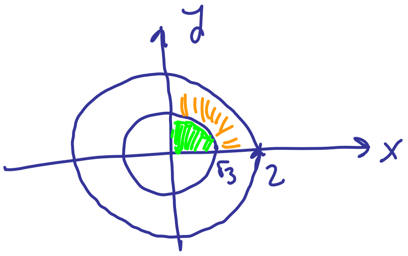
$$= \frac{\pi}{6}\left(1 - e^{-4}\right)$$

Question 2 (24 points) Write (do **not** evaluate) a triple integral to compute the volume of the region that lies inside the sphere $x^2 + y^2 + z^2 = 4$ and between the planes $z = 0$ and $z = 1$ using:

a) Cartesian (rectangular) coordinates.

$$V = \frac{\frac{4}{3}\pi 2^3}{2} - 4\int_0^{\sqrt{3}}\int_0^{\sqrt{3-x^2}}\int_1^{\sqrt{4-x^2-y^2}} 1 \, dz\, dy\, dx$$





b) Cylindrical coordinates.

$$V = 4\int_0^{\pi/2}\int_0^{\sqrt{3}}\int_0^{1} 1 \, r \, dz\, dr\, d\theta + 4\int_0^{\pi/2}\int_{\sqrt{3}}^{2}\int_0^{\sqrt{4-r^2}} 1 . r \, dz\, dr\, d\theta$$

OR

$$V = \frac{\frac{4}{3}\pi 2^3}{2} - 4\int_0^{\pi/2}\int_0^{\sqrt{3}}\int_1^{\sqrt{4-r^2}} 1 \, r \, dz\, dr\, d\theta$$

$$\left.\begin{aligned} x &= \rho \sin\phi \cos\theta \\ y &= \rho \sin\phi \sin\theta \end{aligned}\right\} x^2+y^2 = \rho^2 \sin^2\phi$$

$$\boxed{z = \rho\cos\phi = 1}$$

c) Spherical coordinates.

$$V = 4\int_0^{\pi/2}\int_0^{\pi/3}\int_0^{\boxed{1/\cos\phi}} 1 \; \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$+ \; 4\int_0^{\pi/2}\int_{\pi/3}^{\pi/2}\int_0^{2} 1 \, \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

OR

$$V = \frac{\frac{4}{3}\pi 2^3}{2} - 4\int_0^{\pi/2}\int_0^{\pi/3}\int_{1/\cos\phi}^{2} 1 \, \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

Question 3 (16 points) Let $f(x,y) = (y-x^2)(y-2x^2)$.

a)Show that on any line through the origin, f has a local minimum at $(0,0)$.

$f(x,y) = y^2 - 2x^2y - x^2y + 2x^4 = 2x^4 - 3x^2y + y^2$

Let $y = mx$ be any line through origin.

Restriction of $f(x,y)$ on this line is $f(x, mx) \equiv f(x) = 2x^4 - 3x^2mx + m^2$

$f(x) = 2x^4 - 3mx^3 + m^2x^2$

$f'(x) = 8x^3 - 9mx^2 + 2m^2x = 0 \Rightarrow \boxed{x=0}$ is a critical point

$f''(x) = 24x^2 - 9mx + 2m^2 \Rightarrow f''(0) = 2m^2 > 0 \Rightarrow$ by the 2nd deriv test, $x=0$ is a local min. pt.

Note also that, along y-axis, that is when $x=0$, $f = y^2$ and $y=0$ (that is $(x,y)=(0,0)$) is again a loc.min point.

$\therefore$ Along lines through origin, f has a local minimum at $(0,0)$.

b)Find and classify all critical points of f.

$f(x,y) = y^2 - 2x^2y - x^2y + 2x^4 = 2x^4 - 3x^2y + y^2$

$f_x = 8x^3 - 6xy = 2x(4x^2 - 3y) \overset{①}{=} 0$

$f_y = -3x^2 + 2y \overset{②}{=} 0$

$① \; x = 0 \overset{②}{\Rightarrow} y = 0 \Rightarrow (0,0)$

$① \; x \neq 0 \overset{①}{\Rightarrow} 4x^2 - 3y = 0 \overset{②}{\Rightarrow} \frac{8}{3}y - 3y = 0 \Rightarrow \frac{-1}{3}y = 0 \Rightarrow y = 0 \overset{①}{\Rightarrow} x = \Rightarrow \underline{(0,0)}$

Thus $(0,0)$ is the only critical point.

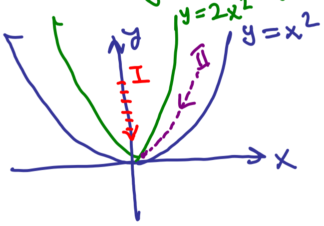
$f_{xx} = 24x^2 - 6y \rightarrow f_{xx}(0,0) = 0$

$f_{yy} = 2 \rightarrow f_{yy}(0,0) = 2$

$f_{xy} = f_{yx} = -6x \rightarrow f_{xy}(0,0) = 0$

$D(0,0) = f_{xx}(0,0)f_{yy}(0,0) - f_{xy}^2(0,0) = 0.2 - 0 = 0$

$\Rightarrow$ TESTFAILS…?

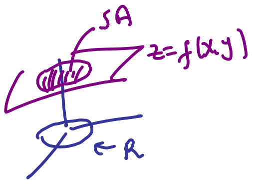


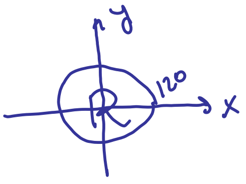
$\underset{(x,y)\in I}{f(x,y) \overset{\nearrow}{=}} f(0,y) = y^2 \Rightarrow y=0$ local min $\Rightarrow$ along I, $f(0,0)$ is a loc.min

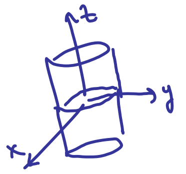
$\underset{\text{along II}}{f(x,y) \underset{\nearrow}{=}} f(x, \frac{3}{2}x^2) = 2x^4 - 3x^2\frac{3}{2}x^2 + \frac{9}{4}x^4 = \frac{17}{4}x^4 - \frac{9}{2}x^4 = \frac{-1}{4}x^4 \Rightarrow$ along II $f(0,0)$ is a loc.MAX

Thus $(0,0)$ is a saddle point.

Question 4 (12 points) Find the area of that part of the hyperbolic paraboloid $z = x^2 - y^2$ that lies inside the cylinder $x^2 + y^2 = 120^2$.







$x^2+y^2=120^2$ in xy-plane $\rightsquigarrow$ in xyz space

$f(x,y) = x^2 - y^2$

$f_x = 2x$

$f_y = -2y$

$1 + f_x^2 + f_y^2 = 1 + 4(x^2+y^2)$

$$SA = \iint_R \sqrt{1 + f_x^2 + f_y^2}\, dA = \iint_R \sqrt{1 + 4(x^2+y^2)}\, dA = \int_0^{2\pi}\int_0^{120} \sqrt{1+4r^2}\, r\, dr\, d\theta$$

$u = 1 + 4r^2$

$du = 8r\,dr$

$$= 2\pi\frac{1}{8}\int_1^{1+4\cdot 120^2} u^{1/2}\, du = \frac{\pi}{4}\frac{2}{3}u^{3/2}\Big|_1^{1+4\cdot120^2} = \frac{\pi}{6}\left(\sqrt{(1+4\cdot120^2)^3} - 1\right)$$ unitsquare.

Question 5 (24 points) Let $z(x,y) = yf(\frac{x}{y^2})$ where f is a continuously differentiable function with $f(1) = 1$ and $f'(1) = 2$.

a) Find a unit vector $\vec{v}$ so that $D_{\vec{v}}z$ at the point $P_0(1,1)$ is minimum.

$D_{\vec{v}}z(P_0) = \nabla z(P_0) \bullet \vec{v} = \|\nabla z(P_0)\| \cos\theta$ is minimum if $\cos\theta = -1$ ie, $\vec{v} = \frac{-\nabla z(P_0)}{\|\nabla z(P_0)\|}$

$z_x(x,y) = y f'(\frac{x}{y^2}) \cdot \frac{1}{y^2} \Rightarrow z_x(P_0) = f'(1) = 2$

$z_y(x,y) = f(\frac{x}{y^2}) + y f'(\frac{x}{y^2})(-2)\frac{x}{y^3} \Rightarrow z_y(P_0) = f(1) + f'(1)(-2) = 1-4 = -3$

thus $\nabla z(P_0) = \langle 2, -3 \rangle$

so $\vec{v} = \frac{-1}{\sqrt{13}} \langle 2, -3 \rangle$.

b) If directional derivative of z in the same direction with $\vec{w} = 2\vec{i} + \vec{j}$ at the point $P_1(2,2)$ is $\frac{1}{\sqrt{5}}$ then find $f(\frac{1}{2})$.

let $\vec{v} = \frac{\vec{w}}{\|\vec{w}\|} = \frac{1}{\sqrt{5}} \langle 2, 1 \rangle$. Then $D_{\vec{v}}z(P_1) = \nabla z(P_1) \bullet \vec{v} = \frac{1}{\sqrt{5}}$ where

$z_x(P_1) = 2 \cdot f'(\frac{1}{2}) \frac{1}{4} = f'(\frac{1}{2}) \frac{1}{2}$, $z_y(P_1) = f(\frac{1}{2}) + 2f'(\frac{1}{2})(-2)\frac{2}{8}$

$= f(\frac{1}{2}) - f'(\frac{1}{2})$

Then

$$\frac{1}{\cancel{\sqrt{5}}} = \langle \tfrac{1}{2}f'(\tfrac{1}{2}), f(\tfrac{1}{2}) - f'(\tfrac{1}{2}) \rangle \bullet \frac{1}{\cancel{\sqrt{5}}} \langle 2, 1 \rangle$$

$$1 = \cancel{f'(\tfrac{1}{2})} + f(\tfrac{1}{2}) - \cancel{f'(\tfrac{1}{2})}$$

Hence $f(\frac{1}{2}) = 1$

c) If $\|\nabla z(2,2)\| = 1$ then find all possible values of $f'(\frac{1}{2})$.

$\nabla z(2,2) = \langle \frac{1}{2}f'(\frac{1}{2}), 1 - f'(\frac{1}{2}) \rangle = \frac{1}{4}(f'(\frac{1}{2}))^2 + 1 + (f'(\frac{1}{2}))^2 - 2f'(\frac{1}{2})$

$= 1$

let $x = f'(\frac{1}{2})$. Then we have

$\frac{5}{4}x^2 - 2x + 1 = 1 \Rightarrow x(\frac{5}{4}x - 2) = 0$

$\Rightarrow x = 0$ or $x = \frac{8}{5}$

$\therefore f'(\frac{1}{2}) \in \{0, \frac{8}{5}\}$

M E T U
Department of Mathematics

Group	CALCULUS II Final Exam	List No.

Code : *Math 120*
Acad. Year : *2009-2010*
Semester : *Spring*
Coordinator : *Muhiddin Uğuz*

Date : *June.03.2010*
Time : *09:30*
Duration : *135 minutes*

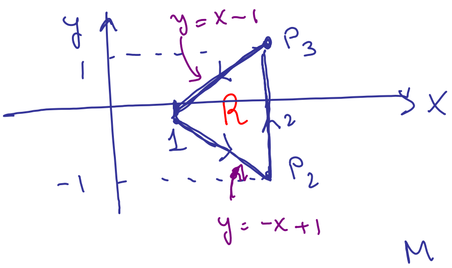
Last Name :
Name : Student No. :
Department : Section :
Signature :

10 QUESTIONS ON 6 PAGES
TOTAL 100 POINTS

1	2	3	4	5	6	7	8	9	10		

S H O W Y O U R W O R K

Question 1 (10 points) Use **Green's theorem** to evaluate the line integral $\oint_C (y+\ln x)dx + (x^3+e^{-y})dy$ where C is the positively oriented boundary of the triangle with vertices $P_1(1,0)$, $P_2(2,-1)$, $P_3(2,1)$.



$\oint_c M\,dx + N\,dy$ where $M(x,y) = y + \ln x$
$N(x,y) = x^3 + e^{-y}$

Note that $N_x - M_y = 3x^2 - 1$

M & N are differentiable everywhere, C is positively oriented,

hence by green's thm, we have

$$\oint_{c=\partial R} M\,dx + N\,dy = \iint_R N_x - M_y \, dA = \int_1^2 \int_{-x+1}^{x-1} 3x^2 - 1 \, dy\, dx = \int_1^2 (3x^2-1)\underbrace{\left[(x-1) - (-x+1)\right]}_{2x-2} dx$$

$$= 2\int_1^2 3x^3 - 3x^2 - x + 1 \, dx = 2\left[\frac{3x^4}{4} - \frac{\cancel{3}x^3}{\cancel{3}} - \frac{x^2}{2} + x\right]_1^2$$

$$= 2\left[\left(3\cdot 4 - 8 - \cancel{2} + \cancel{2}\right) - \left(\frac{3}{4} - \cancel{1} - \frac{1}{2} + \cancel{1}\right)\right] = 2\left[4 - \frac{1}{4}\right]$$

$$= 2\,\frac{15}{4} = 15/2$$

Question 2 (14 points) Determine whether the followings are true of false. If true give a **short explanation**, if false give a **counter example**.

a) If $\sum_{n=1}^{\infty} a_n$ is divergent, then $\sum_{n=1}^{\infty} |a_n|$ is also divergent.

True : We know that if $\sum |a_n|$ is convergent, then so is $\sum a_n$ (absolutely convergent)
that is $P \Rightarrow Q$ or equivalently $\sim Q \Rightarrow \sim P$. In other words
if $\sum |a_n|$ is not conv. that is if $\sum |a_n|$ is div, then $\sum a_n$ is not conv,
that is $\sum a_n$ divergent.

b) For any vectors $\vec{v}$ and $\vec{w}$ in space $\mathbf{R}^3$ we have $(\vec{v}+\vec{w}) \times \vec{v} = \vec{v} \times \vec{w}$.

since $\|\vec v \times \vec v\| = \|\vec v\| \, \|\vec v\| \sin 0 = 0$, we have $\|\vec v \times \vec v\| = 0$, and hence $\vec v \times \vec v = \vec 0$

FALSE

Thus $(\vec v + \vec w) \times \vec v = \underbrace{\vec v \times \vec v}_{= \vec 0} + \vec w \times \vec v = \vec w \times \vec v = \ominus \vec v \times \vec w$

SHOW YOUR WORK

c) The linear equation $120x + 2010y = 0$ represents a line in space $\mathbf{R}^3$.

FALSE

$120x + 2010y = 0$ defines a line in plane, $\mathbb{R}^2$. But it defines
a plane in $\mathbb{R}^3$ with normal vector $\vec n = \langle 120, 2010, 0 \rangle$.

d) If $\vec{F}(t)$ and $\vec{G}(t)$ are differentiable vector functions then $\frac{d}{dt}[\vec{F}(t) \times \vec{G}(t)] = \vec{F}'(t) \times \vec{G}'(t)$.

FALSE

$$\frac{d}{dt}(F \times G) = F' \times G + F \times G'$$

e) If $f(x,y) = \ln y$, then $\nabla f(x,y) = \frac{1}{y}$.

FALSE

$$\nabla f = \left\langle 0, \frac{1}{y} \right\rangle$$

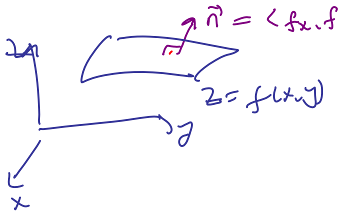
f) Let $f(x,y,z)$ be a differentiable real valued function and C be a smooth oriented curve, then $\int_C f(x,y,z)ds = -\int_{-C} f(x,y,z)ds$.

FALSE

line integral of real valued function does not depend on the orientation the curve C. For line integral of vector valued functions, orientation is important

g) Let Let $f(x,y)$ be a differentiable real valued function and let S be the graph of f, obtained by setting $z = f(x,y)$. Then gradient of f, that is $\nabla f(x_0, y_0)$ is normal (perpendicular) to the tangent plane of S at the corresponding point.

FALSE



$\vec n = \langle f_x, f_y, -1 \rangle = \nabla F$ where

$F(x,y,z) = f(x,y) - z = 0$

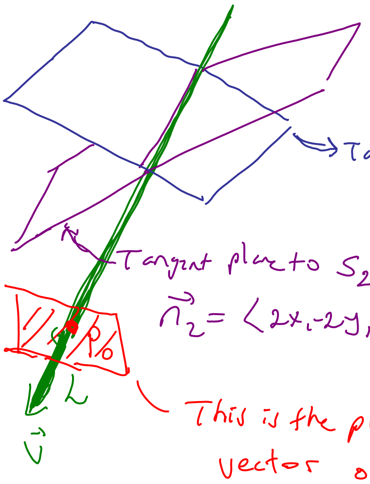
Question 3 (8 points) Let $f(x,y)=\dfrac{x^2-y^2}{x^2+y^2}$. Is it possible to define $f(0,0)$ so that f becomes a continuous function everwhere? Explain.

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=mx}} f(x,y) = \lim_{x\to 0} \frac{x^2-m^2x^2}{x^2+m^2x^2} = \frac{1-m^2}{1+m2}$$ depends on m. Hence

$\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist. To be continuous $f(0,0)$ must be the same as $\lim_{(x,y)\to(0,0)} f(x,y)$. Thus we can not define $f(0,0)$ so that f becomes continuous everywhere.

SHOW YOUR WORK

Question 4 (10 points) Let L be the line of intersection of the tangent planes drawn at $P_0(1,1,0)$ to the surfaces $S_1:\ x^2y+xy^3+z^3+xyz=2$ and $S_2:\ z=x^2-y^2$. Find an equation of the plane P which passes through the point P_0 and which is perpendicular to the line L. (That is direction vector of L and normal vector of P are paralel.)



Tangent plane to S_1 at P_0

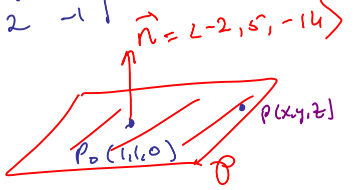
$$\vec{n}_1 = \langle 2xy+y^3+yz,\ x^2+3xy^2+xz,\ 3z^2+xy\rangle\Big|_{P_0} = \langle 3,4,1\rangle$$

Tangent plane to S_2 at P_0

$$\vec{n}_2 = \langle 2x,-2y,-1\rangle\Big|_{P_0} = \langle 2,-2,-1\rangle$$

This is the plane in question, it's normal vector $\vec{n}$ is parallel to $\vec{v}$ = direction vector of L and $\vec{v}$ is parallel to $\vec{n}_1\times\vec{n}_2$

$$\vec{n}_1\times\vec{n}_2 = \begin{vmatrix} i & j & k \\ 3 & 4 & 1 \\ 2 & -2 & -1 \end{vmatrix} = \langle -4+2,\ 2+3,\ -6-8\rangle = \langle -2,5,-14\rangle = \vec{n}$$

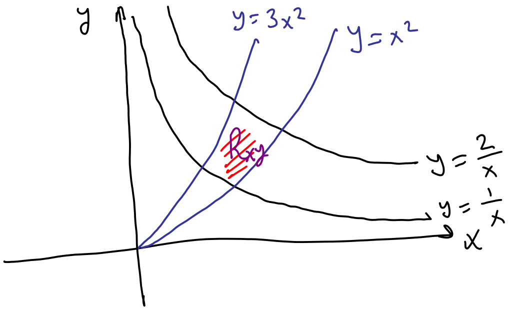


$$P(x,y,z)\in P \iff \langle x-1,\ y-1,\ z\rangle\cdot\langle -2,5,-14\rangle = 0$$

$$-2x+2+5y-5-14z=0$$

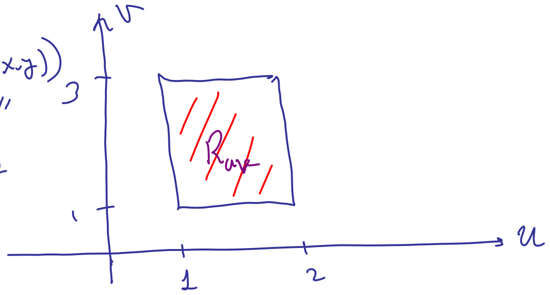
$$-2x+5y-14z=3$$

Question 5 (10 points) Find the area of the region in the first quadrant bounded by the curves $y = \frac{1}{x}$, $y = \frac{2}{x}$, $y = x^2$ and $y = 3x^2$.



$r(x,y) = (u(x,y), v(x,y))$

$u = xy$, $v = \frac{y}{x^2}$



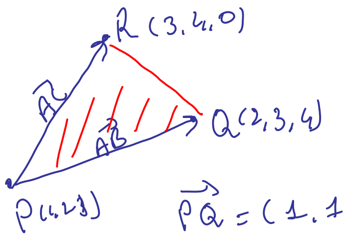
$$\text{Area} = \iint_{R_{xy}} 1\,dy\,dx = \iint_{R_{uv}} 1 \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dv\,du = \int_1^2 \int_1^3 \left|\frac{1}{3v}\right| dv\,du$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}} = \frac{1}{\det\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}} = \frac{1}{\det\begin{bmatrix} y & x \\ \frac{-2y}{x^3} & \frac{1}{x^2} \end{bmatrix}} = \frac{1}{\frac{y}{x^2} + \frac{2\cancel{x}y}{x^{\cancel{3}2}}} = \frac{1}{\frac{3y}{x^2}} = \frac{x^2}{3y} = \frac{1}{3v}$$

$$= \frac{1}{3} \ln v \Big|_1^3 = \frac{1}{3}\left(\ln(3) - \cancel{\ln(1)}^{0}\right) = \frac{\ln 3}{3}$$

SHOW YOUR WORK

Question 6 (8 points) Let $P = (1,2,3)$, $Q = (2,3,4)$, $R = (3,4,0)$, and A be the area of the triangle determined by the points P, Q, R. Find a vector $\vec{v}$ such that $|\vec{v}| = A$ and $\vec{v}$ is perpendicular to the plane determined by the points P, Q, R.



$$\text{Area}(PQR) = \frac{|\vec{PQ} \times \vec{PR}|}{2} = \frac{|(-5,5,0)|}{2} = \frac{\sqrt{50}}{2} = \frac{5\sqrt{2}}{2}$$

Thus $\boxed{\|\vec{v}\| = \frac{5\sqrt{2}}{2}}$

$\vec{PQ} = (1,1,1)$

$\vec{PR} = (2,2,-3)$

and $\vec{v} \,//\, (\vec{PQ} \times \vec{PR}) \Rightarrow \vec{v} = k(-5,5,0)$
$k \in \mathbb{R} \setminus \{0\}$

$\Downarrow$

$$\frac{5\cancel{\sqrt{2}}}{2} = |k| \cancel{\sqrt{50}}$$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 2 & 2 & -3 \end{vmatrix}$$

$$= (-3-2, -(-3-2), 2-2)$$

$$= (-5, 5, 0)$$

$\Rightarrow |k| = \frac{1}{2} \Rightarrow k = \mp\frac{1}{2}$

$\Rightarrow \vec{v} = \mp\frac{1}{2}\langle -5, 5, 0\rangle.$

Name: .... Lastname: .... Student id: .... Signature: ....

Question 7 (10 points) Find the maximum and minimum values of the **sum** of two non-negative numbers x and y which satisfy $x^3 + y^3 + 3x^2y = 15$.

That is; find Max/min of $\left[\begin{array}{l} f(x,y) = x+y \\ g(x,y) = x^3+y^3+3x^2y-15=0 \end{array}\right.$ Subject to

Use Lagrange multipliers method: set $\nabla f = \lambda \nabla g$;

$$f_x = \boxed{\begin{array}{l} 1 \overset{1}{=} \lambda(3x^2+6xy) \\ 1 \overset{2}{=} \lambda(3y^2+3x^2) \\ x^3+y^3+3x^2y \overset{3}{=} 15 \end{array}} \begin{array}{l} = \lambda g_x \\ = \lambda g_y \\ \end{array}$$

$f_y =$

3 equations, 3 unknown ☺

$$\left.\begin{array}{l} 3\lambda x^2 + 6\lambda xy = 1 \\ 3\lambda x^2 + 3\lambda y^2 = 1 \end{array}\right\} \quad \lambda y^2 = 2\lambda xy \xRightarrow[\lambda \neq 0]{} y^2 = 2xy$$

$\lambda \neq 0 \Leftarrow ①$

$y^2 = 2xy \rightarrow y = 0 \overset{③}{\Rightarrow} x^3 = 15 \Rightarrow x = \sqrt[3]{15}$

$y^2 = 2xy \rightarrow y \neq 0 \Rightarrow y = 2x$

$\Downarrow ③$

$x^3 + 8x^3 + 6x^3 = 15$

$x^3 = 1$

$\Downarrow$

$\boxed{x=1} \Rightarrow \boxed{y=2}$

$f(\sqrt[3]{15}, 0) = \sqrt[3]{15} + 0 = \sqrt[3]{15}$ $\underline{\underline{\text{min}}}$

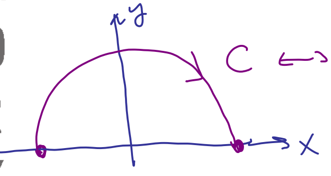
$f(1,2) = 1+2 = 3$ MAX

S
H
O
W

Y
O
U
R

W
O
R
K

Question 8 (10 points) Evaluate $\int_C (2 + x^2y)ds$, where C is the upper half of the unit circle oriented in clockwise (negative) direction.



$C \leftrightarrow r(t) = (\sin t, \cos t)$; $t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

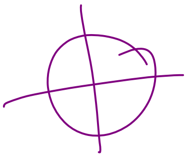
$r'(t) = (\cos t, -\sin t)$

$\|r'(t)\| = 1$

$$\int_{-\pi/2}^{\pi/2} (2 + \sin^2 t \cos t)\, 1\, dt = 2 \cdot \pi + \int_{-\pi/2}^{\pi/2} \sin^2 t \cos t\, dt = 2\pi + \int_{-1}^{+1} u^2 du$$

$u = \sin t$
$du = \cos t\, dt$

$$= 2\pi + \left.\frac{u^3}{3}\right|_{-1}^{+1}$$



$$= 2\pi + \frac{1}{3}(1-(-1)) = 2\pi + \frac{2}{3}$$

(Note that, since this is a line integral of a real valued function $f(x,y) = 2 + x^2y$, orientation of C is <u>not</u> important; that is $\int_C 2+x^2y\, ds = \int_{-C} 2+x^2y\, ds$)

Question 9 (10 points) Find Taylor series of $f(x) = \dfrac{x^3}{(1-x)^2}$ about $x = 0$. Find the interval **I** of convergence of this series.(Hint: $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for all x such that $|x|$ 1.)

$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \Longrightarrow$ Taking derivatives of both sides, we obtain

$\frac{+1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1} \Longrightarrow$ multiply both sides with $+x^3$;

$\frac{x^3}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n+2} = \sum_{n=3}^{\infty} (n-2) x^n$ is the Maclorian Series of $f(x)$ (that is Taylor series about $x=0$)

since radius of convergence of $\sum_{n=0}^{\infty} x^n$ is 1, and this series is obtained by differentiation $\sum x^n$, for $\sum_{n=1}^{\infty} n x^{n+2}$ has also $R = 1$.

Hence $\frac{x^3}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n+2}$ for $x \in (-1, +1)$

at the both end points $x = 1$ and $x = -1$, $\sum n x^{n+1}$ is $\sum \overset{0}{\cancel{\neq}} \textcircled{n}$ div. nth Ter Test

Hence $I = (-1, +1)$.

S
H
O
W

Y
O
U
R

W
O
R
K

Question 10 (10 points) Let $z = z(x, y)$ be a function of x and y defined by the equation $\ln(z \cos y) + x^{(2^y)} = 2y$. Find $z_x(1, 0)$.

$\ln(z\cos y) + x^{(2^y)} = 2y$. Take partial derivative of both sides with respect to x :)

$\frac{1}{z\,\cancel{\cos y}} \cdot z_x \cdot \cancel{\cos y} + 2^y \cdot x^{2^y - 1} = 0$

when $\left.\begin{matrix} x=1 \\ y=0 \end{matrix}\right\}$ $\ln(z\cos 0) + 1^{(2^0)} = 0$

$\overset{\ln z}{e} = \overset{-1}{e}$

$z = \frac{1}{e}$

at $(x, y) = (1, 0)$ $(\Rightarrow z = 1/e)$

$\frac{1}{\frac{1}{e}} \cdot z_x + 2^0 \cdot 1^{2^0 - 1} = 0$

$e \cdot z_x + 1 \cdot 1 = 0$

$z_x = \frac{-1}{e}$

OR $\ln(z\cos y) = 2y - x^{(2^y)} \Longrightarrow z\cos y = e^{2y - x^{(2^y)}} \Longrightarrow z = \frac{e^{2y - x^{(2^y)}}}{\cos y}$

$\Longrightarrow z_x = \frac{1}{\cos y} e^{2y - x^{(2^y)}} \cdot (-2^y)\, x^{(2^y - 1)} \Longrightarrow z_x(1,0) = 1 \cdot e^{-1}(-1) \cdot 1$
$= -1/e$