# M E T U
## Department of Mathematics

| | CALCULUS OF FUNCTIONS OF SEVERAL VARIABLES MidTerm 1 | |
|---|---|---|
| Code : *Math 120*<br>Acad. Year : *2010-2011*<br>Semester : *Spring*<br>Coordinator: *Muhiddin Uğuz*<br>Date : *April.09.2011*<br>Time : *9:30*<br>Duration : *120 minutes* | Last Name :<br>Name : Student No. :<br>Department : Section :<br>Signature : | |
| | 7 QUESTIONS ON 6 PAGES<br>TOTAL 100 POINTS | |

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**Question 1 (15 pts)** Let $\{a_n\}_{n=1}^{\infty}$ be the sequence defined recursively by

$$a_1 = 3$$
$$a_{n+1} = \sqrt{15 + 2a_n}, \ n = 1, 2, 3, \cdots$$

**a)** Show that $\{a_n\}$ is increasing.

Induction on $n$:
- $a_2 = \sqrt{15+2a_1} = \sqrt{15+6} = \sqrt{21} > 3 = a_1$
- Assume $a_n > a_{n-1}$
- $a_{n+1} = \sqrt{15+2a_n} > \sqrt{15+2a_{n-1}} = a_n$

Hence, by induction, $a_n$ is an increasing sequence.

OR

$a_{n+1} - a_n = \sqrt{15+2a_n} - a_n > 0$ (i.e. $a_n$ is increasing) if and only if $\sqrt{15+2a_n} > a_n \iff 15+2a_n > a_n^2 \iff a_n^2 - 2a_n - 15 < 0$
$\iff (a_n+3)(a_n-5) < 0$
$\iff a_n < 5$ (since $a_n$'s are all positive)

Hence if we can prove $a_n < 5$ $\forall n=1,2,\ldots$ we can say that $a_n$'s are increasing.

**b)** Show that $\{a_n\}$ is bounded.

$a_1 = 3 > 0$, $a_n$'s are defined in terms of square roots, and hence $a_n \geq 0$ $\forall n$.

Also as proven in (*), $a_n < 5$ $\forall n=1,2,\ldots$

Thus $|a_n| < 5$ $\forall n=1,2,\ldots$

(*) Claim: $a_n < 5$ $\forall n=1,2,\ldots$

Proof: By induction on $n$
- $n=1 \Rightarrow a_1 = 3 < 5$ true
- $n=k \Rightarrow$ assume $a_k < 5$
- $n=k+1 \Rightarrow a_{k+1} = \sqrt{15+2a_k} < \sqrt{15+10} = 5$.

**c)** Does $\lim_{n\to\infty} a_n$ exist? Explain. If yes, find it.

Since $a_n$ is an increasing sequence, which is bounded from above, $\lim_{n\to\infty} a_n$ exists. Moreover if $\lim_{n\to\infty} a_n = L$ then also $\lim_{n\to\infty} a_{n+1} = L$. Hence taking limit of both sides of the equation $a_{n+1} = \sqrt{15+2a_n}$; we obtain

$L = \sqrt{15+2L} \Rightarrow L^2 - 2L - 15 = (L-5)(L+3) = 0.$

Since $a_n > 0$ $\forall n$, $L \neq -3$. Thus $\lim_{n\to\infty} a_n = 5$

**Question 2 (24 pts)**

Determine whether given series converge or diverge. Give reasons for your answers.

a) $\sum_{n=1}^{\infty} n^2 \sin(\frac{1}{n^2+1})$

Apply $n^{th}$ Term Test: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} n^2 \sin\left(\frac{1}{n^2+1}\right) =$

$$= \lim_{n\to\infty} n^2 \cdot \frac{\sin\left(\frac{1}{n^2+1}\right)}{\left(\frac{1}{n^2+1}\right)} \cdot \left(\frac{1}{n^2+1}\right)$$

Let $m = \frac{1}{n^2+1}$. Then $n^2 = \frac{1}{m} - 1 = \frac{1-m}{m}$

$$= \lim_{m\to 0} \frac{\sin m}{m} \cdot \frac{m \cdot (1-m)}{m} = 1$$

($\frac{\sin m}{m} \to 1$, $(1-m) \to 1$)

since $\lim_{n\to\infty} a_n = 1 \neq 0$, $\Sigma a_n$ diverges

b) $\sum_{n=1}^{\infty} \frac{3n+7}{8n^3+n^2+1}$

let $a_n = \frac{3n+7}{8n^3+n^2+1}$ and $b_n = \frac{3}{8n^2}$. Notice that both $a_n > 0$ & $b_n > 0$ $\forall n$

So we can use limit comparison test:

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{x\to\infty} \frac{3x+7}{8x^3+x^2+1} \cdot \frac{8x^2}{3} = 1$, we have $\Sigma a_n \Leftrightarrow \Sigma b_n$ that is either both conv. or both div.

As $\Sigma b_n = \frac{3}{8} \Sigma \frac{1}{n^2}$ is convergent (by p-test)

we have $\Sigma a_n$ is convergent (by limit Comp. T.)

c) $\sum_{n=1}^{\infty} \frac{n!}{1 \cdot 3 \cdot 5 \cdots (2n+1)} = \Sigma a_n$.

Notice that $a_n > 0$ ; hence we can use Ratio Test:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!}{1.3.5\cdots(2n+1).(2n+3)} \cdot \frac{1.3\cdots(2n+1)}{n!}$$

$= \lim_{n\to\infty} \frac{n+1}{2n+3} = \frac{1}{2} < 1$. Thus by Ratio Test, $\Sigma a_n$ is convergent.

d) $\sum_{n=1}^{\infty} \frac{\cos n}{n^3} = \Sigma a_n$

Notice that $0 \leq |a_n| = \left|\frac{\cos n}{n^3}\right| \leq \frac{1}{n^3}$ $\forall n = 1, 2, \ldots$

since $\Sigma \frac{1}{n^3}$ is convergent (by p-test), using comparison test we conclude that $\Sigma |a_n|$ is convergent, and hence $\Sigma a_n$ is (absolutely) convergent.

**Question 3** (18 pts)

**a)** Write down an equation of the line $L$ which is the intersection of the planes:

$$\pi_1: \ x+y+z=3 \quad \text{and} \quad \pi_2: \ 2x+3y+6z=12.$$

Notice that $P_0(0,2,1)$ and $P_1(-3,6,0)$ are two points on both of the planes, hence are two points on $L$. Therefore we can take $\vec{P_1P_0} = \vec{P_0} - \vec{P_1} = (3,-4,1)$ as direction vector of $L$.

(alternatively we can take $\vec{n_1} \times \vec{n_2} = (1,1,1) \times (2,3,6)$ as direction vector of $L$)

$$\left.\begin{aligned} x &= 0 + 3t \\ y &= 2 - 4t \\ z &= 1 + 1t \end{aligned}\right\} \text{ parametric Eqn. of } L$$

$t \in \mathbb{R}$

OR $\dfrac{x-0}{3} = \dfrac{y-2}{-4} = \dfrac{z-1}{1}$ $(=t)$ — symmetric (standard) equation of $L$

OR $\vec{P} = (x,y,z) = (3t, 2-4t, 1+t)$, $t \in \mathbb{R}$ — vector equation of $L$

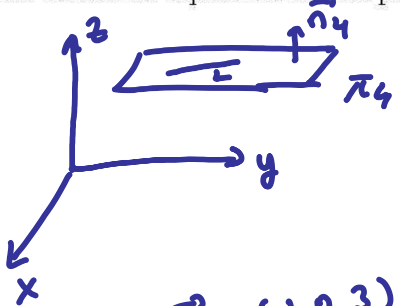
**b)** Find the point $P$ at which the line $L$ intersects the plane $\pi_3: \ x+y-z=0$.

$P$ has coordinates $(x,y,z)$ such that $x+y-z=0$ and $x=3t$, $y=2-4t$, $z=1+t$ for some $t \in \mathbb{R}$. Thus we have

$$3t + 2 - 4t - 1 - t = 0 \implies 1 = 2t \implies t = \tfrac{1}{2}$$

$\implies (x,y,z) = \left(\frac{3}{2}, 2-2, 1+\frac{1}{2}\right) \implies P$ has coordinates $\left(\frac{3}{2}, 0, \frac{3}{2}\right)$

**c)** Write down an equation of the plane $\pi_4$ containing the line $L$ and parallel to the $y$-axis.



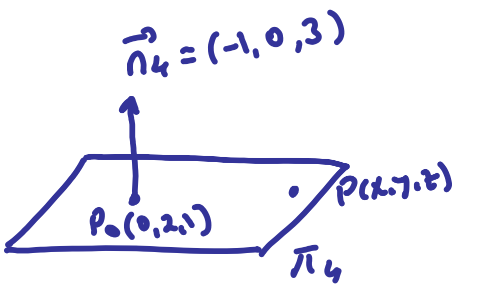

$\vec{n_4} \perp (3,-4,1)$ ← direction vector of $L$

and

$\vec{n_4} \perp (0,1,0)$

$\Rightarrow n_4$ is parallel to $(3,-4,1) \times (0,1,0) = (-1,0,3)$

$$\det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix} = (-1,0,3)$$




$P(x,y,z) \in \pi_4 \iff \vec{P_0P} \cdot \vec{n_4} = 0$

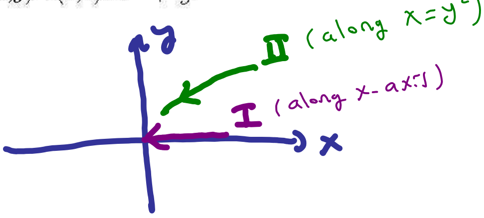
$(x, y-2, z-1) \cdot (-1,0,3) = 0$

$-x + 3z - 3 = 0$

$\boxed{3z - x = 3}$ — equation of $\pi_4$

**Question 4 (12 pts)** Determine whether the following limits exist or not. Explain.

a) $\lim\limits_{(x,y)\to(0,0)} \dfrac{2xy^2}{x^2+y^4}$




• $\lim\limits_{\substack{(x,y)\to(0,0)\\ \text{along I}}} f(x,y) = \lim\limits_{\substack{x\to 0\\ y=0}} f(x,0) = \lim\limits_{x\to 0} \dfrac{0}{x^2} = \lim\limits_{x\to 0} 0 = 0$ (Thus, if limit exists, it is 0)

• $\lim\limits_{\substack{(x,y)\to(0,0)\\ \text{along II}}} f(x,y) = \lim\limits_{y\to 0} f(y^2,y) = \lim\limits_{y\to 0} \dfrac{2y^4}{y^4+y^4} = \lim\limits_{y\to 0} 1 = 1$

since $0 \neq 1$, limit does not exist.

b) $\lim\limits_{(x,y)\to(0,0)} y\sin\left(\dfrac{1}{\sqrt{x^2+y^2}}\right)$

$0 \le \left| y \sin \dfrac{1}{\sqrt{x^2+y^2}} \right| \le |y|$

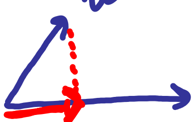
as $(x,y)\to(0,0)$ both sides $\to 0$

By squeezing them, we have $\lim\limits_{(x,y)\to(0,0)} |f(x,y)| = 0$

But this implies, (again by squeezing them) $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$.

**Question 5 (8 pts)** Write the vector $\vec{u} = 3\vec{j} - 3\vec{k}$ as the sum of a vector parallel to $\vec{v} = -\vec{i} - \vec{j}$ and a vector orthogonal (that is, perpendicular) to $\vec{v}$.

$\vec{u} = (0,3,-3) = \text{proj}_{\vec{v}}\vec{u} + (\vec{u} - \text{proj}_{\vec{v}}\vec{u})$



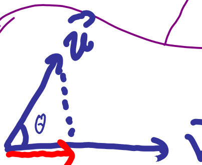
$\vec{v} = (-1,-1,0)$

$\vec{u}\cdot\vec{v} = 0 - 3 + 0 = -3$

$\|\vec{v}\|^2 = 1 + 1 + 0 = 2$

$\text{proj}_{\vec{v}}\vec{u} = \dfrac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v} = \dfrac{-3}{2}\vec{v} = \boxed{\dfrac{3}{2}(1,1,0) = \vec{A}}$

$\vec{u} - \vec{A} = \langle -\tfrac{3}{2}, \tfrac{3}{2}, -3\rangle = -\tfrac{3}{2}\langle 1,-1,2\rangle$




$\text{proj}_{\vec{v}}\vec{u} = \|\vec{u}\|\cos\theta \dfrac{\vec{v}}{\|\vec{v}\|}$ (unit vector in correct direction) $= \|\vec{u}\|\|\vec{v}\|\cos\theta \dfrac{\vec{v}}{\|\vec{v}\|^2} = \dfrac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v}$

$\cos\theta = \dfrac{\|\text{proj}_{\vec{v}}\vec{u}\|}{\|\vec{u}\|}$

Thus

$\vec{u} = (0,3,-3) = \underbrace{\tfrac{3}{2}\langle 1,1,0\rangle}_{\text{a vector parallel to } \vec{v}} + \underbrace{-\tfrac{3}{2}\langle 1,-1,2\rangle}_{\text{a vector orthogonal to } \vec{v}}$

OR

$\vec{u} = \langle 0,3,-3\rangle = \vec{a} + \vec{b}$

where ① $\vec{a} \parallel \langle -1,-1,0\rangle = \vec{v}$

② $\vec{b} \perp \langle -1,-1,0\rangle = \vec{v}$

① ⇒ $\vec{a} = \langle -k,-k,0\rangle$

② ⇒ $\vec{u}\cdot\vec{v} = \vec{a}\cdot\vec{v} + \vec{b}\cdot\vec{v}$ ($\vec{b}\cdot\vec{v} = 0$)

$-3 = k + k \Rightarrow k = -3/2$

Thus $\vec{a} = \tfrac{3}{2}\langle 1,1,0\rangle$

$\vec{b} = \vec{u} - \vec{a} = -\tfrac{3}{2}\langle 1,-1,2\rangle$

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**Question 6 (8 pts)**

Determine the interval of convergence of the power series $\sum_{n=0}^{\infty} \frac{n}{n^2+1}\left(\frac{x-5}{3}\right)^n$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{n+1}{(n+1)^2+1} \frac{|x-5|^{n+1}}{3^{n+1}} \cdot \frac{(n^2+1)}{n} \frac{3^n}{|x-5|^n}$$

$$= \frac{|x-5|}{3} \lim_{n\to\infty} \left(\frac{n+1}{n}\right)\left(\frac{n^2+1}{n^2+2n+2}\right) = \frac{1}{3}|x-5|$$

Thus given power series is convergent if $\frac{1}{3}|x-5| < 1$

That is $|x-5| < 3$

$-3 < x-5 < 3 \Rightarrow 2 < x < 8$

Now check the end points:

$x=2 \Rightarrow \sum_{n=0}^{\infty} \frac{n}{n^2+1} \frac{(-3)^n}{3^n} = \sum_{n=0}^{\infty} (-1)^n \frac{n}{n^2+1}$ ($= a_n$) is an alternating series

since $\lim_{n\to\infty} a_n = 0$ and $a_n$ is decreasing

$\left(\frac{a_{n+1}}{a_n} < 1 \text{ (or } a_{n+1} - a_n < 0 \text{ ; or } f(x) = \frac{x}{x^2+1} \text{ is decreasing)}\right)$

it is a convergent alternating series.

$x = 8 \Rightarrow \sum_{n=0}^{\infty} \frac{n}{n^2+1} \frac{3^n}{3^n} = \sum_{n=0}^{\infty} \frac{n}{n^2+1}$ is a divergent series

(by limit comparison test

$\left(\lim_{n\to\infty} \frac{\frac{n}{n^2+1}}{\frac{1}{n}} = 1 \text{ \& } \sum \frac{1}{n} \text{ div}\right)$

Thus

Interval of convergence is $[2, 8)$

**Question 7 (15 pts)**

**a)** Find the Taylor series expansion of $f(x) = \dfrac{1}{(4-3x)^2}$ about $a = 0$.

Recall that $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \forall\ |x| < 1$

$$\frac{1}{4-3x} = \frac{1}{4}\left(\frac{1}{1-\frac{3}{4}x}\right) = \frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{3}{4}x\right)^n \quad \forall\ \left|\frac{3}{4}x\right| < 1$$

That is

$$\frac{1}{4-3x} = \frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n x^n \quad \forall\ |x| < \frac{4}{3}$$

Take derivative of both sides;

$$\frac{3}{(4-3x)^2} = \frac{1}{4}\sum_{n=1}^{\infty}\left(\frac{3}{4}\right)^n \cdot n\, x^{n-1} \quad \forall\ |x| < \frac{4}{3}$$

Hence $\dfrac{1}{(4-3x)^2} = \dfrac{1}{12}\sum_{n=1}^{\infty}\left(\dfrac{3}{4}\right)^n \cdot n \cdot x^{n-1} \quad \forall\ |x| < \dfrac{4}{3}$

**b)** Determine the radius and interval of convergence of the series you obtained.

$R = \frac{4}{3}$ as obtained above (when we take derivative of a convergent series term by term, R does not change)

To find I, we need to check end points:

$$x = \frac{4}{3} \Rightarrow \frac{1}{12}\sum_{n=1}^{\infty}\left(\frac{3}{4}\right)^n \cdot n \cdot \left(\frac{4}{3}\right)^{n-1} = \frac{1}{12}\sum_{n=1}^{\infty}\frac{3}{4}\cdot n \not\to 0$$

(divergent by nth T.T.)

$$x = -\frac{4}{3} \Rightarrow \frac{1}{12}\sum_{n=1}^{\infty}\left(\frac{3}{4}\right)^n n \left(-\frac{4}{3}\right)^{n-1} = \frac{1}{12}\sum \frac{3}{4}\cdot n(-1)^{n-1} \not\to 0 \quad \text{divergent}$$

Thus $I = \left(-\frac{4}{3}, \frac{4}{3}\right)$

**c)** Find the $120^{th}$ derivative $(f^{(120)})(0)$ of $f$ at the point $x = 0$.

$$f(x) = \frac{1}{(4-3x)^2} = \frac{1}{12}\sum_{n=1}^{\infty}\left(\frac{3}{4}\right)^n \cdot n \cdot x^{n-1} \quad \forall\ |x| < \frac{4}{3}$$

$$= \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = \sum_{n=1}^{\infty}\frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1}$$

Thus

$$\frac{f^{(n-1)}(0)}{(n-1)!} = \frac{1}{12}\left(\frac{3}{4}\right)^n \cdot n \Rightarrow f^{(120)}(0) = \frac{1}{12}\left(\frac{3}{4}\right)^{121} \cdot 121 \cdot 120!$$

$$= \frac{1}{12}\cdot\left(\frac{3}{4}\right)^{121} \cdot 121!$$

# M E T U
## Department of Mathematics

| | CALCULUS OF FUNCTIONS OF SEVERAL VARIABLES<br>MidTerm 2 | |
|---|---|---|
| Code : *Math 120*<br>Acad. Year : *2010-2011*<br>Semester : *Spring*<br>Coordinator : *Muhiddin Uğuz*<br>Date : *April.30.2011*<br>Time : *9:30*<br>Duration : *120 minutes* | Last Name :<br>Name :<br>Department :<br>Signature : | Student No. :<br>Section : |
| | 6 QUESTIONS ON 4 PAGES<br>TOTAL 100 POINTS | |

| 1 | 2 | 3 | 4 | 5 | 6 | SHOW YOUR WORK | |
|---|---|---|---|---|---|---|---|
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**Question 1 (6+7+7=20 pts)** Let $F(x, y, z) = xy + ye^z$.

a)Compute the gradient $\nabla F$ at $P(0, 1, 0)$.

$\nabla F = \langle F_x, F_y, F_z \rangle = \langle y, x + e^z, ye^z \rangle$

$\Rightarrow \nabla F(P) = \langle 1, 1, 1 \rangle$

b)Compute the directional derivative of $F$ at the point $P(0, 1, 0)$ in the direction of the vector $\vec{u} = -\vec{i} + \vec{j} + \vec{k}$.

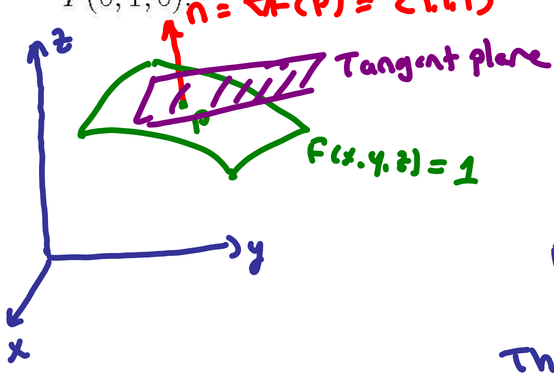
$\vec{v} = \frac{\vec{u}}{\|\vec{u}\|} = \frac{1}{\sqrt{3}} \langle -1, 1, 1 \rangle$ is the unit vector in direction of $\vec{u}$

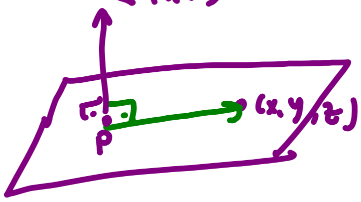
Directional derivative of F, in direction of $\vec{u}$ is by definition $D_{\vec{v}}F(P)$ and defined as $D_{\vec{v}}F(P) = \lim_{t \to 0} \frac{F(\vec{P} + t\vec{v}) - F(\vec{P})}{t}$ (if limit exists)

But, since F is a polynomial (and hence differentiable), this limit is equal to $\nabla F(P) \cdot \vec{v}$.

Hence $D_{\vec{v}}F(P) = \nabla F(P) \cdot \vec{v} = \langle 1, 1, 1 \rangle \cdot \frac{1}{\sqrt{3}} \langle -1, 1, 1 \rangle = \frac{1}{\sqrt{3}}$

c)Write an equation of the tangent plane to the (level) surface $F(x, y, z) = 1$ at the point $P(0, 1, 0)$. $\vec{n} = \nabla F(P) = \langle 1, 1, 1 \rangle$



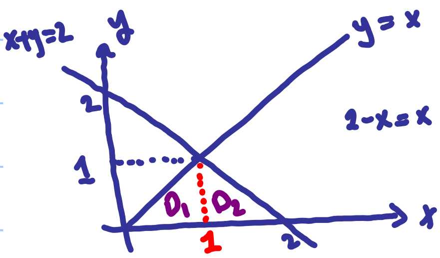




$P(x, y, z)$ is a point on the tangent plane iff $\langle 1, 1, 1 \rangle \cdot \langle x - 0, y - 1, z - 0 \rangle = 0$

Thus $x + y + z = 1$ is an equation of the tangent plane at P.

**Question 2 (6+6+8=20 pts)** Consider the double integral $\int_D\int (x+2y)\,dA$ where $D$ is the region in $\mathbb{R}^2$ bounded by the lines $y=0$, $y=x$, and $x+y=2$.

a) Write the double integral (do **not** evaluate) as an iterated integral in the order $dxdy$.




$2-x=x \Rightarrow x=1$

$$\iint_D (x+2y)\,dA = \int_0^1\int_y^{2-y}(x+2y)\,dx\,dy$$

b) Write the double integral (do **not** evaluate) as an iterated integral in the order $dydx$.

$$\iint (x+2y)\,dA = \int_0^1\int_0^x (x+2y)\,dy\,dx + \int_1^2\int_0^{2-x}(x+2y)\,dy\,dx$$

$D = D_1 \cup D_2$

c) Evaluate $\int_D\int (x+2y)\,dA$.

From part (a); $\int_0^1\int_y^{2-y}(x+2y)\,dx\,dy = \int_0^1 \frac{x^2}{2}+2xy\Big|_{x=y}^{x=2-y}\,dy$

$$= \int_0^1 \frac{(2-y)^2}{2} + 2(2-y)y - \frac{y^2}{2} - 2y^2\,dy = \int_0^1 2-2y+\frac{y^2}{2}+4y-2y^2-\frac{y^2}{2}-2y^2\,dy$$

$$= \int_0^1(-4y^2+2y+2)\,dy = -\frac{4y^3}{3}+y^2+2y\Big|_0^1 = -\frac{4}{3}+1+2 = \frac{5}{3}$$

**Question 3 (10 pts)** The equation $z^3 - zy^2 + yx = 3$ defines $z$ implicitly as a function of $x$ and $y$. It is given that $z=2$ when $(x,y)=(-3,1)$. Evaluate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at $(-3,1)$.

- Taking partial derivative of both sides of the given equation w.r.t $x$; we obtain: $3z^2 z_x - z_x y^2 + y = 0 \Rightarrow z_x = \frac{-y}{3z^2-y^2}$

  $\Rightarrow$ at $(x,y,z)=(-3,1,2)$ we have

  $z_x = \frac{-1}{11}$

- Taking partial derivative of both sides of the given equation w.r.t $y$; we obtain: $3z^2 z_y - z_y\cdot y^2 - 2zy + x = 0 \Rightarrow z_y = \frac{2zy-x}{3z^2-y^2}$

  $\Rightarrow$ at $(x,y,z)=(-3,1,2)$ we have

  $z_y = \frac{7}{11}$

**Question 4** (9+15=24 pts)

a) Find and classify the critical points of the function $f(x,y) = \frac{1}{3}x^3 - \frac{1}{4}x + y^2$.

$(x_0, y_0)$ is a critical point of $f(x,y)$ if $f_x(x_0,y_0) = 0 = f_y(x_0,y_0)$.

$f_x(x,y) = x^2 - \frac{1}{4} = 0 \Rightarrow x = \mp 1/2$

$f_y(x,y) = 2y = 0 \Rightarrow y = 0$

$(x_1, y_1) = (-1/2, 0)$ & $(x_2, y_2) = (1/2, 0)$ are the only critical points.

$f_{xx}(x,y) = 2x = A$

$f_{yy}(x,y) = 2 = C$

$f_{xy}(x,y) = 0 = B$

$\Rightarrow \Delta = B^2 - AC = 0 - 4x = -4x$

- at $(-\frac{1}{2}, 0)$: $\Delta = -4 \cdot (-\frac{1}{2}) = 2 > 0 \Rightarrow (-\frac{1}{2}, 0)$ saddle point
- at $(1/2, 0)$: $\Delta = -4 (\frac{1}{2}) = -2 < 0$ & $A = 2 \cdot \frac{1}{2} = 1 > 0 \Rightarrow (\frac{1}{2}, 0)$ is a local minimum point

b) Using Lagrange Multiplier method on the boundary, find the absolute maximum and minimum values of $f(x,y)$ given in part (a) on the region

$$D = \{(x,y) \in \mathbf{R}^2 : (x-1)^2 + 4y^2 \le 4\}.$$

We have checked the critical points inside $D$ in part (a).

Now let's check the boundary points of $D$ ($\partial D = \{(x,y) \in \mathbb{R}^2 : (x-1)^2 + 4y^2 = 4\}$)

i.e. Find Max/min of $f(x,y) = \frac{1}{3}x^3 - \frac{1}{4}x + y^2$ subject to $g(x,y) = (x-1)^2 + 4y^2 - 4 = 0$ — use Lagrange multiplier method.

We need to solve $\nabla f = \lambda \nabla g$:

$$x^2 - \frac{1}{4} \overset{1}{=} 2\lambda(x-1)$$
$$2y \overset{2}{=} 8\lambda y$$
$$(x-1)^2 + 4y^2 - 4 \overset{3}{=} 0$$

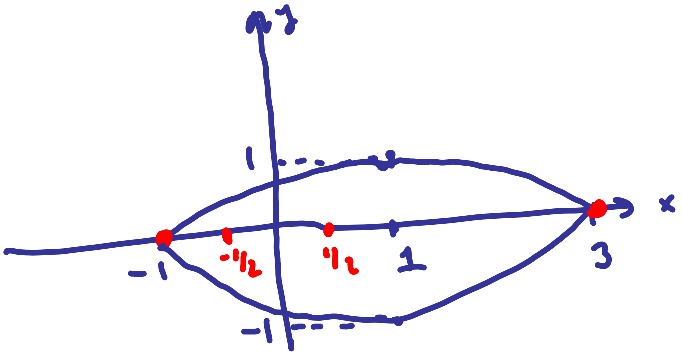
3 unknowns $(x, y, \lambda)$, 3 equations

$y = 0 \overset{3}{\Rightarrow} (x-1)^2 = 4 \Rightarrow x - 1 = \mp 2$: $x_1 = -1 \Rightarrow (-1, 0)$; $x_2 = 3 \Rightarrow (3, 0)$

$y \neq 0 \overset{2}{\Rightarrow} 2 = 8\lambda \Rightarrow \lambda = \frac{1}{4} \overset{1}{\Rightarrow} x^2 - \frac{1}{4} - \frac{1}{2}x + \frac{1}{2} = 0 \Rightarrow x^2 - \frac{1}{2}x + \frac{1}{4} = 0 \Rightarrow$

$4x^2 - 2x + 1 = 0$

$\Delta = 4 - 4 \cdot 4 \cdot 1 < 0 \Rightarrow$ no solution



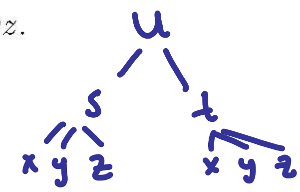

Note that since $f$ is a continuous function on the closed domain $D$, it has absolute max & absolute min on $D$.

$f(\frac{1}{2}, 0) = \frac{-1}{12}$, $f(-1, 0) = \frac{-1}{12}$ } absolute minimum

$f(3, 0) = \frac{33}{4}$ } absolute maximum

**Question 5 (9+5=14 pts)**

Let $U = F(s,t)$ have partial derivatives where $s(x,y,z) = xy$ and $t(x,y,z) = yz$.




a)Show that $x\dfrac{\partial U}{\partial x} - y\dfrac{\partial U}{\partial y} + z\dfrac{\partial U}{\partial z} = 0$.

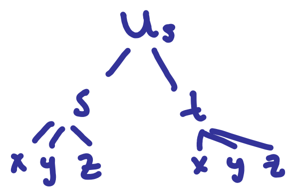
$$U_x = U_s \cdot S_x + U_t \cdot t_x = U_s \cdot y + U_t \cdot 0$$

$$U_y = U_s \cdot S_y + U_t \cdot t_y = U_s \cdot x + U_t z$$

$$U_z = U_s \cdot S_z + U_t \cdot t_z = U_s \cdot 0 + U_t \cdot y$$

Hence

$$x U_x - y U_y + z U_z = xy U_s - xy U_s - yz U_t + yz U_t = 0$$




b)Write $\dfrac{\partial^2 U}{\partial y \partial x} = U_{xy}$ in terms of partial derivatives of $F(s,t)$.

$$U_{xy} = (U_x)_y = (U_s \cdot y)_y = (U_s)_y \cdot y + U_s$$

$$= [U_{ss} \cdot S_y + U_{st} \cdot t_y]\, y + U_s$$

$$= U_{ss} xy + U_{st} yz + U_s = xy F_{ss} + yz F_{st} + F_s$$

**Question 6 (4+4+4=12 pts)** Let $f(x,y) = \begin{cases} \frac{x}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

a)Compute $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ for $(x,y) \neq (0,0)$.

$$f_x(x,y) = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \quad \forall (x,y) \neq (0,0)$$

$$f_y(x,y) = \frac{-2yx}{(x^2+y^2)^2} \quad \forall (x,y) \neq (0,0)$$

b)Compute $\dfrac{\partial f}{\partial y}(0,0)$.

$$f_y(0,0) = \lim_{t \to 0} \frac{f(0,0+t) - f(0,0)}{t} = \lim_{t \to 0} \frac{\frac{0}{0+t^2} - 0}{t} = \lim_{t \to 0} \frac{0}{t} = \lim_{t \to 0} 0 = 0$$

c)Does $\dfrac{\partial f}{\partial x}(0,0)$ exist?

$$\lim_{t \to 0} \frac{f(0+t,0) - f(0,0)}{t} = \lim_{t \to 0} \frac{\frac{t}{0+t^2} - 0}{t} = \lim_{t \to 0} \frac{1}{t^2} \to +\infty$$

$\therefore$ $f_x(0,0)$ does not exist.

# M E T U
## Department of Mathematics

| CALCULUS OF FUNCTIONS OF SEVERAL VARIABLES<br>Final Exam | |
|---|---|
| Code : *Math 120*<br>Acad. Year : *2010-2011*<br>Semester : *Spring*<br>Coordinator: *Muhiddin Uğuz*<br><br>Date : *June.02.2011*<br>Time : *9:30*<br>Duration : *150 minutes* | Last Name :<br>Name : Student No. :<br>Department : Section :<br>Signature :<br><br>8 QUESTIONS ON 6 PAGES<br>TOTAL 100 POINTS |

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | SHOW YOUR WORK | |
|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | |

**Question 1 (12 pts)** Test the following series for convergence.

a) $\sum_{n=1}^{\infty} \frac{n^\pi}{\pi^n} = \sum_{n=1}^{\infty} a_n$ . Note that $a_n = \frac{n^\pi}{\pi^n} > 0 \ \forall n$ . Hence we can use

Ratio Test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^\pi}{\pi^{n+1}} \cdot \frac{\pi^n}{n^\pi} = \frac{1}{\pi} \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^\pi$

$= \frac{1}{\pi} < 1.$ Thus $\sum a_n$ is convergent

b) $\sum_{n=1}^{\infty} e^{\frac{1}{\pi^n}} = \sum a_n.$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} e^{\frac{1}{\pi^n}} = e^{\lim_{n\to\infty} \frac{1}{\pi^n}} = e^0 = 1 \neq 0.$

($e^x$ is continuous $\forall x$)

since $\lim_{n\to\infty} a_n \neq 0$, by $n^{th}$ term test $\sum a_n$ is divergent.

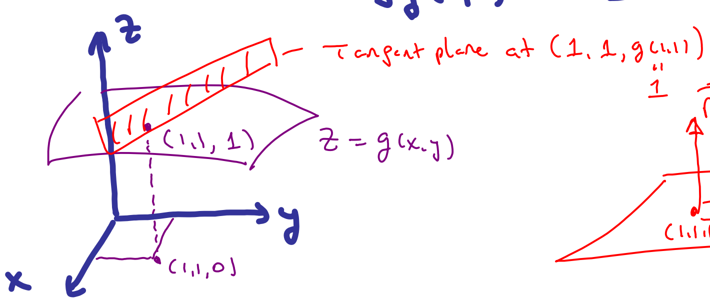
**Question 2 (10 pts)** Let $g(x,y) = f(xf(\frac{x}{y}))$ where $f : \mathbb{R} \to \mathbb{R}$ continuously differentiable function with $f(1) = 1$ and $f'(1) = 1$. Find the equation of the tangent plane to the graph of $z = g(x,y)$ at the point $(1,1)$.

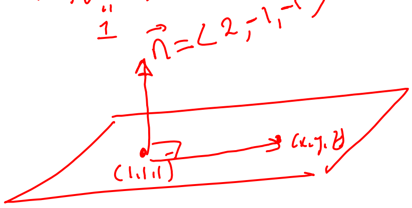
$g_x(x,y) = f'(xf(\frac{x}{y})) \cdot [1 f(\frac{x}{y}) + x f'(\frac{x}{y})\frac{1}{y}]$

$\Rightarrow g_x(1,1) = 1 \cdot [1 + 1] = 2$

$g_y(x,y) = f'(xf(\frac{x}{y})) \cdot x \cdot f'(\frac{x}{y}) \cdot \frac{-x}{y^2}$

$\Rightarrow g_y(1,1) = -1$




$(x,y,z) \in$ Tangent plane iff

$\langle x-1, y-1, z-1 \rangle \cdot \langle 2,-1,-1 \rangle = 0$

$2x - y - z = 2 - 1 - 1 = 0$

$2x - y - z = 0$

**Question 3 (12 pts)**

a) Find the Taylor series about $c = 0$ for the function if $f(x) = \int_0^x \frac{1 - cost}{t} dt$.

(Hint: You may use the fact that $cost = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n)!}$ for all $t \in \mathbb{R}$.)

since $\cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots$

we have

$$\frac{1-\cos t}{t} = \frac{t}{2!} - \frac{t^3}{4!} + \frac{t^5}{6!} - \frac{t^7}{8!} + \cdots \quad \forall t \in \mathbb{R}.$$

Taking the integral of both sides (we can take integral of right hand side "term by term" since it is a convergent series $\forall t \in \mathbb{R}$) we obtain:

$$\int_0^x \frac{1-\cos t}{t} dt = \frac{x^2}{2 \cdot 2!} - \frac{x^4}{4 \cdot 4!} + \frac{x^6}{6 \cdot 6!} \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n}}{(2n)\cdot(2n)!}$$

b) Use part (a) to evaluate $f(1)$ correct to 3 decimal places, i.e. with an error less than 0.0005.

Since the series we obtained above is a convergent alternating series, if we take the sum of first k terms as an approximation to the infinite sum, we know that error is less than absolute value of the $(k+1)^{th}$ term.

First of all $0.0005 = \frac{5}{10.000} = \frac{1}{2.000}$

$$\frac{1}{4.4!} = \frac{1}{96} < \frac{1}{2000} \text{ no} \quad , \quad \frac{1}{6.6!} = \frac{1}{4320} < \frac{1}{2000} \text{ yes} \checkmark$$

Hence

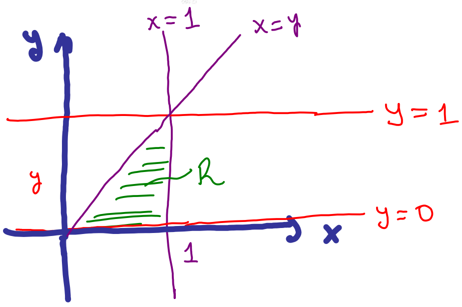
$$f(1) = \int_0^1 \frac{1-\cos t}{t} dt = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{(2n)\cdot(2n)!}$$

$$\approx \frac{1}{2.2!} - \frac{1}{4.4!} \quad \left(\text{error} < \frac{1}{6.6!} < \frac{1}{2000} = 0.0005\right)$$

**Question 4 (18 pts)** Compute the following double integrals.

a) $\int_0^1 \int_y^1 \sin(\pi x^2)\,dx\,dy.$

$$= \iint_R \sin(\pi x^2)\,dA = \int_0^1 \int_0^x \sin(\pi x^2)\,dy\,dx$$




$$= \int_0^1 x \sin(\pi x^2)\,dx = \frac{1}{2\pi}\int_0^\pi \sin u\,du$$

$u = \pi x^2$

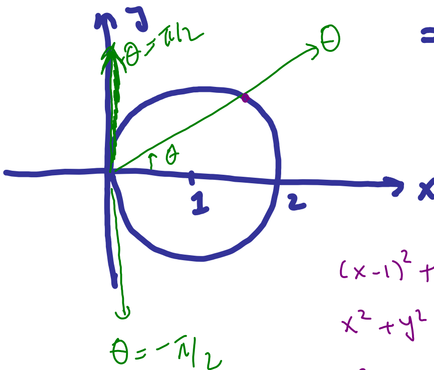
$du = 2\pi x\,dx$

$$= \frac{-1}{2\pi} \cos u \Big|_0^\pi$$

$$= \frac{-1}{2\pi}[-1-1] = \frac{1}{\pi}$$

b) $\int\int_\Omega (x^2+y^2)^{1/2} dA$, where $\Omega = \{(x,y) \in \mathbb{R}^2 |\ (x-1)^2 + y^2 \le 1\}$.

$$= \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} (r^2)^{1/2} \cdot r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2} \frac{r^3}{3}\Big|_0^{2\cos\theta} d\theta$$




$$= \frac{8}{3}\int_{-\pi/2}^{\pi/2} \cos^3\theta\,d\theta = \frac{8}{3}\int_{-\pi/2}^{\pi/2} (1-\sin^2\theta)\cos\theta\,d\theta$$

$u = \sin\theta$

$du = \cos\theta\,d\theta$

$(x-1)^2 + y^2 = 1$

$x^2 + y^2 - 2x + \cancel{1} = \cancel{1}$

$r^2 - 2r\cos\theta = 0$

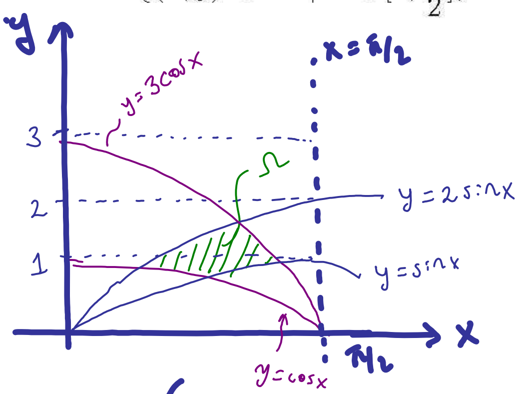
$r(r - 2\cos\theta) = 0$

$$= \frac{8}{3}\int_{-1}^{+1} (1-u^2)\,du = \frac{8}{3}\left[u - \frac{u^3}{3}\right]_{-1}^{+1}$$

$$= \frac{8}{3}\left[\left(1-\frac{1}{3}\right) - \left(-1+\frac{1}{3}\right)\right] = \frac{8}{3}\cdot\frac{4}{3} = \frac{32}{9}$$

c) $\int\int_\Omega \frac{\tan x}{y^3} dA$, where

$\Omega = \{(x,y) \in \mathbb{R}^2 \mid x \in [0, \frac{\pi}{2}],\ \sin x \le y \le 2\sin x,\ \cos x \le y \le 3\cos x\}$.



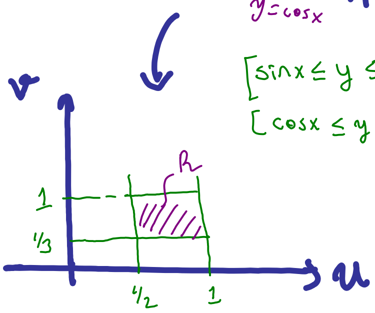

Let $u = \frac{\sin x}{y}$, $v = \frac{\cos x}{y}$

Then

$dx\,dy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$ where

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{\left|\frac{\partial(u,v)}{\partial(x,y)}\right|} = \frac{1}{\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}} = \frac{1}{\begin{vmatrix} \frac{\cos x}{y} & \frac{-\sin x}{y^2} \\ \frac{-\sin x}{y} & \frac{-\cos x}{y^2} \end{vmatrix}} = \frac{1}{\left|\frac{-\cos^2 x - \sin^2 x}{y^3}\right|} = |y^3| = y^3$$

$y > 0$ on $\Omega$




$[\sin x \le y \le 2\sin x] \Rightarrow [1 \le \frac{y}{\sin x} \le 2] \Rightarrow [\frac{1}{2} \le u \le 1]$

$[\cos x \le y \le 3\cos x] \Rightarrow [1 \le \frac{y}{\cos x} \le 3] \Rightarrow [\frac{1}{3} \le v \le 1]$

Hence $\iint_\Omega \frac{\tan x}{y^3}\,dA = \iint_\Omega \tan x \left(\frac{1}{y^3}\,dy\,dx\right)$ $du\,dv$

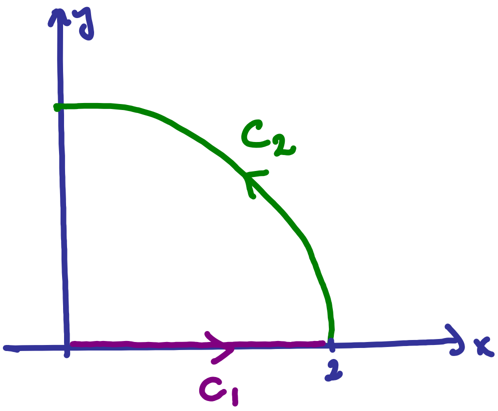
$$= \iint_R \frac{u}{v}\,du\,dv = \int_{1/3}^1 \int_{1/2}^1 \frac{u}{v}\,du\,dv$$

$$= \int_{1/3}^1 \frac{1}{v}\frac{u^2}{2}\Big|_{1/2}^1 dv = \int_{1/3}^1 \frac{1}{v}\frac{1}{2}\left(1-\frac{1}{4}\right)dv = \frac{3}{8}\ln|v|\Big|_{1/3}^1$$

$$= \frac{3}{8}\left(\cancel{\ln 1}^{\,0} - \ln\frac{1}{3}\right) = \frac{3\ln 3}{8}$$

**Question 5 (12 pts)** Let $C = C_1 \cup C_2$ be the union of two curves from $(0,0)$ to $(0,2)$ where $C_1$ is the **straight line** segment from the point $(0,0)$ to the point $(2,0)$ and $C_2$ is the **part of the circle** with radius 2 from the point $(2,0)$ to the point $(0,2)$.

**a)** Find a parametrization for $C_1$ and a parametrization for $C_2$.




$C_1 \leftrightarrow r_1(t) = (t, 0) \quad : \quad t \in [0, 2]$

$C_2 \leftrightarrow r_2(t) = (2\cos t, 2\sin t) \; : \; t \in [0, \pi/2]$

**b)** Evaluate $\int_C (y-x)dx + 2xdy.$ $= \int_{C_1} (y-x)\,dx + 2x\,dy + \int_{C_2} (y-x)\,dx + 2x\,dy$

on $C_1$: $dx = dt$, $dy = 0 \cdot dt$ ; on $C_2$: $dx = -2\sin t\,dt$, $dy = 2\cos t\,dt$

$= \int_0^2 (0-t)\,dt + \int_0^{\pi/2} \left[(2\sin t - 2\cos t)(-2\sin t) + 2\cdot 2\cos t \cdot 2\cos t\right] dt$

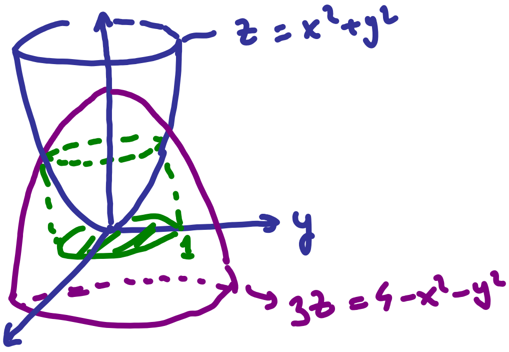
$\cos 2x = \cos^2 x - \sin^2 x$
$= 2\cos^2 x - 1$
$= 1 - 2\sin^2 x$
$\Rightarrow \cos^2 x = \frac{1}{2}(1+\cos 2x)$
$\sin^2 x = \frac{1}{2}(1-\cos 2x)$

$= -\frac{t^2}{2}\Big|_0^2 + \int_0^{\pi/2} \left[-4\sin^2 t + 4\sin t\cos t + 8\cos^2 t\right] dt$

$= -2 - \frac{4}{2}\int_0^{\pi/2} 1-\cos 2t\,dt + 4\int_0^{\pi/2} \sin t\cos t\,dt + \frac{8}{2}\int_0^{\pi/2} 1+\cos 2t\,dt$

$= -2 - 2\left[\pi/2\right] + 4\left(\sin\frac{\pi}{2} - \sin 0\right) + 4\left[\frac{\pi}{2}\right] = -2 - \pi + 4 + 2\pi = 2 + \pi$

**Question 6 (12 pts)** Find the volume of the solid lying between the paraboloids $z = x^2 + y^2$ and $3z = 4 - x^2 - y^2$.




$3(x^2+y^2) = 4 - x^2 - y^2 \Rightarrow x^2 + y^2 = 1$

$V = \iint \left[\frac{(4-x^2-y^2)}{3} - (x^2+y^2)\right] dA$

$= \int_0^{2\pi}\int_0^1 \frac{4}{3}(1-r^2)\,r\,dr\,d\theta$

$= 2\pi \cdot \frac{4}{3}\left[\frac{r^2}{2} - \frac{r^4}{4}\right]_0^1$

$= \frac{8\pi}{3}\left[\frac{1}{2} - \frac{1}{4}\right] = \frac{2\pi}{3}$

Surname:..............Name:..............Student Id:.............Signature:..............

## Question 7 (10 pts)

**a)** Find the constant $A$ so that the integral $\oint_C Aydx + 120xdy$ is **zero** for any simple closed curve $C$.

Given integral is zero for any simple closed curve C means the vector field $\langle Ay, 120x \rangle$ is conservative.

Thus we must have $\frac{\partial}{\partial y}(Ay) = \frac{\partial}{\partial x}(120x)$

That is $A = 120$

**b)** Find the constant $B$ so that the integral $\oint_C Bydx + 120xdy$ gives the **area** enclosed by $C$, where $C$ is any positively oriented simple closed curve. Explain.

Recall that if R is a bounded closed simply connected region with positively oriented boundary curve $\partial R = C$, then

$$\text{Area}(R) = \iint_R 1\, dA = \iint_R (N_x - M_y)\, dA = \oint_{\partial R = C} M dx + N dy$$

Hence to get Area, we need $M(x,y) = B \cdot y$, $N(x,y) = 120x$ satisfy $N_x - M_y = 1$

But $N_x = 120$, $M_y = B$ $\Big\}$ $120 - B = 1 \Rightarrow B = 119$

**Question 8 (14 pts)** Let $C$ be the curve of intersection of the plane $2z + x + y = 2$ and the paraboloid $z = x^2 + y^2$. Using Lagrange multipliers method to find the points on $C$ that are nearest to and farthest from origin.

We are asked to find Max/min of $d(x,y,z) = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2}$ (or equivalently Max/min of $f(x,y,z) = x^2 + y^2 + z^2$, and then take square root), subject to two constraints:

That is find Max/min of $f(x,y,z) = x^2 + y^2 + z^2$

subject to $g(x,y,z) = x + y + 2z - 2 = 0$

and $h(x,y,z) = x^2 + y^2 - z = 0$

set $\nabla f = \lambda \nabla g + \mu \nabla h$ and solve the system:

$$\left.\begin{aligned} 2x &\overset{1}{=} \lambda + 2\mu x \\ 2y &\overset{2}{=} \lambda + 2\mu y \\ 2z &\overset{3}{=} 2\lambda - \mu \\ x + y + 2z &\overset{4}{=} 2 \\ x^2 + y^2 - z &\overset{5}{=} 0 \end{aligned}\right]$$

5 unknowns $(x, y, z, \lambda, \mu)$, 5 linear equations, so can be solved.

$(1)-(2) \Rightarrow (x-y) = \mu(x-y) \rightarrow (x-y)(\mu-1) = 0$

Case $\mu = 1$:

$\Downarrow(1)$ $2x = \lambda + 2x$ $\Downarrow$ $\lambda = 0$ $\Downarrow(3)$ $2z = -1$ $\Downarrow$ $[z = -1/2]$ $\Downarrow(5)$ $x^2 + y^2 = -1/2$ which is impossible. Hence no solution for $\mu = 1$.

Case $\mu \neq 1$:

$\Downarrow$ $x = y$ $\Downarrow(4)$

$(5)\left\{\begin{aligned} x + z &= 1 \\ 2x^2 &= z \end{aligned}\right] \Rightarrow 2x^2 + x - 1 = 0$

$\Delta = 1 - 4 \cdot 4 \cdot (-1) = 9$

$x_1 = \frac{-1+3}{4} = \frac{1}{2} = y_1$

$x_2 = \frac{-1-3}{4} = -1 = y_2$

$z_1 = 2x_1^2 = 2\frac{1}{4} = \frac{1}{2}$

$z_2 = 2x_2^2 = 2 \cdot 1 = 2$

$P_1(x_1, y_1, z_1) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$

$P_2(x_2, y_2, z_2) = (-1, -1, 2)$

$f(P_1) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4} \rightarrow d(P_1) = \frac{\sqrt{3}}{2} = \sqrt{3} \cdot \frac{1}{2}$ minimum

$f(P_2) = 1 + 1 + 4 = 6 \rightarrow d(P_2) = \sqrt{6} = \sqrt{3} \cdot \sqrt{2}$ MAXIMUM.