M E T U
Department of Mathematics

Group	CALCULUS WITH ANALYTIC GEOMETRY II MidTerm 1	List No.

Code : *Math 120* Acad. Year : *2011-2012* Semester : *Spring* Coordinator: *Muhiddin Uğuz* Date : *April.07.2012* Time : *9:30* Duration : *120 minutes*	Last Name : Name : Student No. : Department : Section : Signature : 6 QUESTIONS ON 6 PAGES TOTAL 100 POINTS

1	2	3	4	5	6	SHOW YOUR WORK	

Question 1 (6+6=12 pts)

The sequence of partial sums for the series $\sum_{n=1}^{\infty} a_n$ is $\{s_n\}_{n=1}^{\infty}$ where $s_n = 3 - \frac{n}{2^n}$.

a) Find a_n

Since $S_n = a_1 + a_2 + \cdots + a_{n-1} + a_n$ and

$-\quad S_{n-1} = a_1 + a_2 + \cdots + a_{n-1}$, we have

$$S_n - S_{n-1} = a_n = \left(\cancel{3} - \frac{n}{2^n}\right) - \left(\cancel{3} - \frac{n-1}{2^{n-1}}\right) = -\frac{n}{2^n} + \frac{2n-2}{2^n}$$

$$\Rightarrow \quad a_n = \frac{n-2}{2^n} \quad \forall n = 1, 2, \ldots$$

b) Determine if $\sum_{n=1}^{\infty} a_n$ is convergent. EXPLAIN. If convergent, find the sum.

$\sum_{n=1}^{\infty} a_n$ is defined as $\lim_{n\to\infty} S_n$. Hence

since limit on right hand side exists

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(3 - \frac{n}{2^n}\right) = \lim_{x\to\infty} 3 - \frac{x}{2^x}$$

since each limit exists

L'Hopi's Rule

$$= 3 - \lim_{x\to\infty} \frac{x}{e^{x\ln 2}} = 3 - \lim_{x\to\infty} \frac{1}{\ln 2 \cdot e^{x\ln 2}}$$

$$= 3 - 0 = 3$$

Thus

$$\sum_{n=1}^{\infty} a_n = 3 \quad \text{Convergent}$$

Question 2 (7+7+7=21 pts) Determine whether the following series are convergent or not. Give your explanations.

a) $\sum_{n=1}^{\infty} \frac{n^3}{3^n}$. $= \sum_{n=1}^{\infty} a_n$ where $a_n = \frac{n^3}{3^n} > 0 \ \forall n = 1,2..$ $\Rightarrow$ we can use Ratio Test

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^3}{\cancel{3^{n+1}}} \cdot \frac{\cancel{3^n}}{n^3} = \frac{1}{3} \left(\lim_{n\to\infty} \frac{n+1}{n} \right)^3 = \frac{1}{3} < 1$ Hence by ratio test, given series is convergent

$f(x)=x^3$ is continuous

OR use root test

$$\lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} \frac{n^{3/n}}{3} = \frac{1}{3} \lim_{x\to\infty} x^{\frac{3}{x}} = \frac{1}{3} \lim_{x\to\infty} e^{\frac{3}{x}\ln x} = \frac{1}{3} e^{\lim_{x\to\infty} \frac{3\ln x}{x}}$$

e^x is continuous

$$\overset{\text{L'Hopi Rule}}{=} \frac{1}{3} e^{\lim_{x\to\infty} \frac{3}{x}} = \frac{1}{3} e^0 = \frac{1}{3} < 1 \Rightarrow \sum a_n \text{ conv. by root test}$$

b) $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{\sqrt{n}}$. $= \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} (-1)^n \cdot a_n$ where $a_n = \frac{1}{\sqrt{n}}$

- This is an alternating series
- $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$
- $\frac{a_{n+1}}{a_n} = \frac{\frac{1}{\sqrt{n+1}}}{\frac{1}{\sqrt{n}}} = \sqrt{\frac{n}{n+1}} < 1 \Rightarrow a_{n+1} < a_n \Rightarrow a_n$ is decreasing

$\frac{n}{n+1} \in (0,1)$

Thus by Alternating Series test, $\sum (-1)^n a_n$ is convergent

c) $\sum_{n=1}^{\infty} \sqrt{n} \arcsin \frac{1}{\sqrt{n}}$. $= \sum_{n=1}^{\infty} a_n$

since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt{n} \arcsin \frac{1}{\sqrt{n}} = \lim_{n\to\infty} \frac{\arcsin(\frac{1}{\sqrt{n}})}{(\frac{1}{\sqrt{n}})}$

$= \lim_{m\to 0} \frac{\arcsin(m)}{m} = \lim_{x\to 0} \frac{\arcsin x}{x}$ $\left(\frac{0}{0} \text{ type}\right)$

L'Hopi Rule

$\overset{\text{L'H}}{=} \lim_{x\to 0} \frac{1}{\sqrt{1-x^2}} = 1 \neq 0$, by n^{th} term test, $\sum a_n$ is divergent

Question 3 (6+7+7=20 pts)

Given $\sin x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \quad \forall x \in \mathbb{R}$,

a) Using the above formula for $\sin(x)$, find the Taylor series expansion for $\cos(x^2)$ about $x_0 = 0$.

$$\cos x = \frac{d}{dx}\sin x = \frac{d}{dx}\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!} = \sum_{n=}^{\infty}\frac{(-1)^n}{(2n+1)!}\frac{d}{dx}x^{2n+1}$$

$$= \sum_{n=}^{\infty}\frac{(-1)^n}{(2n+1)!}(2n+1)x^{2n} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

since above formula is true $\forall x \in \mathbb{R}$, we can substitude x^2 in place of x

Hence

$$\cos(x^2) = \sum_{n=0}^{\infty}(-1)^n\frac{x^{4n}}{(2n)!} = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \cdots \quad \forall x \in \mathbb{R}$$

b) Find the 68^{th} derivative $\frac{d^{68}}{dx^{68}}\cos(x^2)$ at the point $x_0 = 0$ **using part (a)**.

since $f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$ (provided that limit of error term is zero),

$\overset{(*)}{=} \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{4n}$, coefficient of x^{68} must be $\frac{f^{(68)}(0)}{68!}$

((*): by uniqueness of the power series representation of $f(x)$ at $x=0$)

since $68 = 4.17$, we have $\frac{(-1)^{17}}{(34)!} = \frac{f^{(68)}(0)}{(68)!}$

Thus $f^{(68)}(0) = -\frac{68!}{34!} = -68 \cdot 67 \cdot \cdots \cdot 35$

c) Find $\cos(\frac{1}{4})$ correct within $|\text{error}| < \frac{1}{1000}$ **using part (a)**.

$$\cos\frac{1}{4} = \cos\left(\frac{1}{2}\right)^2 = \sum_{n=0}^{\infty}(-1)^n\frac{\left(\frac{1}{2}\right)^{4n}}{(2n)!}$$

since this is a convergent alternating series, and $|a_n| = \frac{1}{2^{4n}\cdot(2n)!}$ is decreasing, as an approximation, if we take first n-terms, we know that $|\text{error}| < |a_{n+1}|$

$$= 1 - \frac{\left(\frac{1}{2}\right)^4}{2!} + \frac{\left(\frac{1}{2}\right)^8}{4!} - \frac{\left(\frac{1}{2}\right)^{12}}{6!} + \cdots$$

since $\frac{1}{2^4\cdot 2!} = \frac{1}{2^5} = \frac{1}{32} \not< \frac{1}{1000}$, but $\frac{1}{2^8\cdot 4!} = \frac{1}{256\cdot 24} < \frac{1}{1000}$,

$$\cos\frac{1}{4} \approx 1 - \frac{\left(\frac{1}{2}\right)^4}{2!} = 1 - \frac{1}{32} = \frac{31}{32} \quad \left(|\text{error}| < \frac{1}{256\cdot 4\cdot 3\cdot 2} < \frac{1}{1000}\right)$$

Question 4 (6+10=16 pts) Let

$\mathbb{P}$ be the plane $x + 2y - z - 3 = 0$ and

L be the line $\langle x, y, z\rangle = \langle 1, 1, 0\rangle + t\langle 0, 1, 2\rangle, \quad t \in \mathbb{R}$.

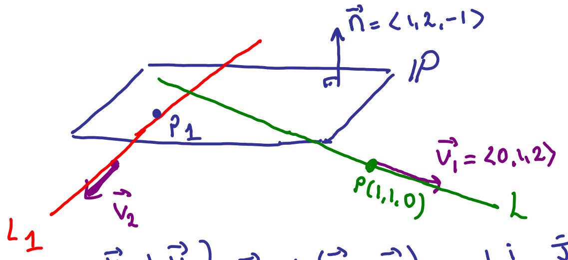
a) Prove that L lies on $\mathbb{P}$

Any point $P(x, y, z)$ on L has coordinates $P(1, 1+t, 2t)$ for some $t \in \mathbb{R}$, and coordinates of these points satisfy the equation of the plane. Because

$$1(1) + 2(1+t) + (-1)(2t) - 3 = 1 + 2 + 2t - 2t - 3 = 0$$

Hence any point $P(x, y, z) \in L$ is also on $\mathbb{P}$.

b) Find a parametric equation of the line L_1 which lies on $\mathbb{P}$, passing through the point $P_1(0, 0, -3)$ and perpendicular to L.



Notice that $P_1(0, 0, -3)$ is on $\mathbb{P}: x + 2y - z = 3$

$(0 + 2.0 - (-3) = 3 \checkmark)$

$$\left.\begin{array}{l} \vec{v}_2 \perp \vec{v}_1 \\ \vec{v}_2 \perp \vec{n} \end{array}\right\} \vec{v}_2 \parallel (\vec{v}_1 \times \vec{n}) = \begin{vmatrix} i & j & k \\ 0 & 1 & 2 \\ 1 & 2 & -1 \end{vmatrix} = \langle -5, 2, -1\rangle$$

Thus we can take $\vec{v}_2 = \langle 5, -2, 1\rangle$ as a direction vector of line L_1 that passes through the point $P_1(0, 0, -3)$.

Hence vector parametric equation of L_1 is

$$\langle x, y, z\rangle = \langle 0, 0, -3\rangle + t\,\langle 5, -2, 1\rangle$$
$$= \langle 5t, -2t, -3+t\rangle \; ; \; t \in \mathbb{R}$$

or, scalar parametric equation of L_1 is :

$$\begin{cases} x = 5t \\ y = -2t \\ z = -3 + t \\ t \in \mathbb{R} \end{cases}$$

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Question 5 (6+4+6=16 pts)

a) At what points of the plane is the function

$$f(x,y) = \begin{cases} \frac{y^3}{x^2+y^2} & : \text{if } (x,y) \neq (0,0) \\ 1 & : \text{if } (x,y) = (0,0) \end{cases}$$

continuous? Explain your answer.

if $(x,y) \neq (0,0)$, $f(x,y)$ is given as a ratio of polynomials in two variables with nonzero denominator, hence f is continuous $\forall (x,y) \neq (0,0)$

at $(0,0)$; $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{y^3}{x^2+y^2} = 0$ by squeezing Thm.

$$0 \leq \left|\frac{y^3}{x^2+y^2}\right| = |y| \frac{1}{x^2+y^2} \leq |y|$$

as $(x,y)\to(0,0)$

0

since $f(0,0) = 1 \neq 0 = \lim_{(x,y)\to(0,0)} f(x,y)$, f is not continuous at $(0,0)$

b) Find the limit $\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2+y^8}$ along the curves $y = kx$ where k is a constant real number.

$$\lim_{\substack{(x,y)\to(0,0) \\ \text{along } y=kx}} \frac{xy^4}{x^2+y^8} = \lim_{x\to 0} \frac{x(kx)^4}{x^2+(kx)^8} = k^4 \cdot \lim_{x\to 0} \frac{x^5}{x^2+k^8x^8}$$

$$= k^4 \cdot \lim_{x\to 0} \frac{x^3}{1+k^8x^6} = \frac{0}{1+0} = \frac{0}{1} = 0$$

c) Does the limit $\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2+y^8}$ exist? If it exists find the limit, otherwise explain why it does not exist.

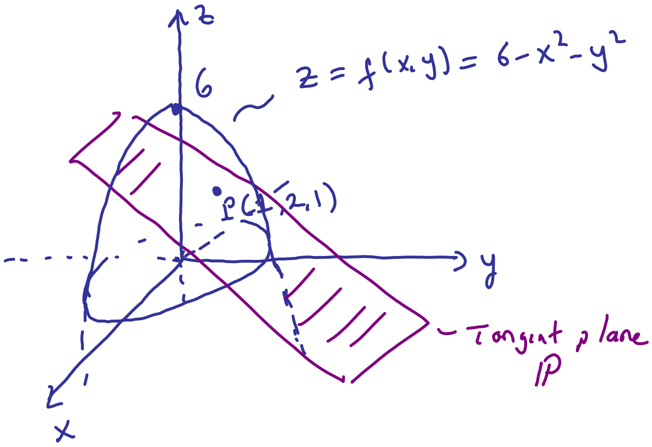
Let's check the limit along the curve $x = y^4$ (which passes through $(0,0)$)

$$\lim_{\substack{(x,y)\to(0,0) \\ \text{along } x=y^4}} f(x,y) = \lim_{y\to 0} f(y^4, y)$$

$$= \lim_{y\to 0} \frac{y^4 \cdot y^4}{y^8+y^8} = \lim_{y\to 0} \frac{1}{1+1} = \frac{1}{2} \neq 0$$

since limits along $y=kx$ and limit along $x=y^4$ are different, limit does not exist.

Question 6 (15 pts) Let $f(x, y) = 6 - x^2 - y^2$. Find a vector equation (parametric equation) and standard (symmetric) equations of the line contained in the tangent plane of the graph of f at the point $P(1, 2, 1)$ and parallel to the xz-plane.

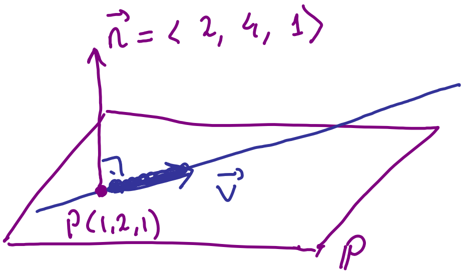


$f_x(x,y) = -2x \Rightarrow f_x(P) = -2$

$f_y(x,y) = -2y \Rightarrow f_y(P) = -4$

$\Rightarrow \vec{n} \parallel \langle -2, -4, -1 \rangle$

$\Rightarrow$ $\vec{n} = \langle 2, 4, 1 \rangle$ can be taken as a normal vector of the tangent plane

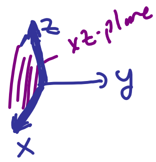


L: through the point $P(1,2,1)$ and with direction vector $\vec{v}$ such that

$\vec{v} \perp \vec{n}$ and $\underbrace{\vec{v} \parallel xz\text{-plane}}$

$\Downarrow$

$\vec{v} \perp \langle 0,1,0 \rangle$



$$\Rightarrow \quad \vec{v} \parallel \vec{n} \times \langle 0,1,0 \rangle = \begin{vmatrix} i & j & k \\ 2 & 4 & 1 \\ 0 & 1 & 0 \end{vmatrix} = \langle -1, 0, 2 \rangle$$

$\therefore$ L is the line through the point $P(1,2,1)$, with direction vector $\vec{v} = \langle -1, 0, 2 \rangle$

Hence we can write

$$P(x,y,z) \in L \iff \langle x, y, z \rangle = \langle 1, 2, 1 \rangle + t \langle -1, 0, 2 \rangle \quad t \in \mathbb{R}$$

"vector parametric equation of L"

or equivalently

$$\left.\begin{aligned} x &= 1 - t \\ y &= 2 \\ z &= 1 + 2t \end{aligned}\right\} \quad \text{"scalar parametric equation of } L\text{"}$$

for $t \in \mathbb{R}$

or equivalently

$$\frac{x-1}{-1} = \frac{z-1}{2} \; ; \; y = 2$$

"standard (symmetric) equation of L"

Alternative solution: Recall that $\langle 1, 0, f_1(1,2) \rangle$ is a vector perpendicular to normal vector of the tangent plane, and parallel to xz-plane. So it can be taken as direction vector of L

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Group	CALCULUS WITH ANALYTIC GEOMETRY II MidTerm 2	List No.

Code : *Math 120* Acad. Year : *2011-2012* Semester : *Spring* Coordinator : *Muhiddin Uğuz* Date : *May.05.2012* Time : *9:30* Duration : *120 minutes*	Last Name : Name : Student No. : Department : Section : Signature : 7 QUESTIONS ON 4 PAGES TOTAL 100 POINTS

1	2	3	4	5	6	7	**SHOW YOUR WORK**	

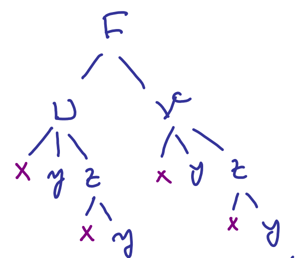
Question 1 (12 pts) Assume that F has continuous partial derivatives and the equation $F(x^2 - z^2, y^2 + xz) = 0$ defines $z = z(x, y)$ as a function of x and y with continuous partial derivatives around the point P_0.

a) Calculate $\dfrac{\partial z}{\partial x}(P_0)$.

$F(x^2-z^2, y^2+xz) = 0$. Let $u = x^2 - z^2 \Rightarrow u_x = 2x,\ u_y = 0,\ u_z = -2z$

$v = y^2 + xz \Rightarrow v_x = z,\ v_y = 2y,\ v_z = x$

Then



To get the term z_x, let's take derivative of F with respect to x, using chain rule;

$$F_u(u,v)\cdot \underbrace{u_x(x,y,z)}_{=2x} + F_u(u,v)\cdot \underbrace{u_z(x,y,z)}_{=-2z}\cdot z_x(x,y) + F_v(u,v)\cdot \underbrace{v_x(x,y,z)}_{=z} + F_v(u,v)\cdot \underbrace{v_z(x,y,z)}_{=x}\, z_x(x,y) = 0$$

Thus
$$z_x(x,y) = \frac{2x\cdot F_u(u,v) + z\, F_v(u,v)}{2z\, F_u(u,v) - x\, F_v(u,v)} = \frac{2x F_1(x^2-z^2, y^2+xz) + z F_2(x^2-z^2, y^2+xz)}{2z\, F_1(x^2-z^2, y^2+xz) - x\, F_2(x^2-z^2, y^2+xz)}$$

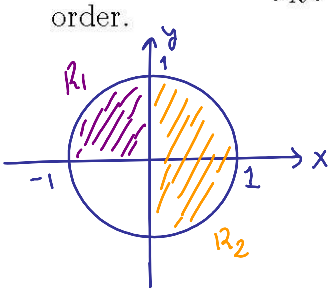
b) If $P_0 = (1, 2)$, and $z(1, 2) = 3$, using appropriate of the values $F_1(1, 2) = 4$, $F_1(-8, 7) = 5$, $F_2(1, 2) = 6$, $F_2(-8, 7) = 7$, find $\dfrac{\partial z}{\partial x}(P_0)$.

$$z_x(1,2) = \frac{2\cdot 1\cdot F_1(1^2-3^2, 2^2+1.3) + 3\, F_2(1^2-3^2, 2^2+1.3)}{2.3\ F_1(1^2-3^2, 2^2+1.3) - 1\ F_2(1^2-3^2, 2^2+1.3)} = \frac{2\,F_1(-8,7) + 3\,F_2(-8,7)}{6\,F_1(-8,7) - F_2(-8,7)}$$

$$= \frac{2\cdot 5 + 3\cdot 7}{6\cdot 5 - 7} = \frac{31}{23}$$

Question 2 (15 pts)

Let R be the region $R = \{(x,y) \in \mathbb{R}^2; x^2 + y^2 \leq 1, x \geq 0 \text{ or } y \geq 0\}$. Express the double integral $\int_R \int f(x,y)dA$ as iterated integrals in Cartesian coordinates in $dydx$ order.



$$\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$$

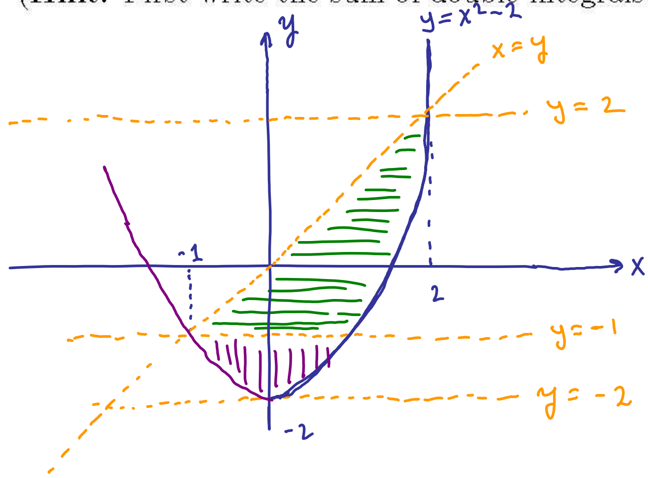
$R = R_1 \sqcup R_2$

$$= \int_{-1}^{0} \int_{0}^{\sqrt{1-x^2}} f(x,y)\,dy\,dx + \int_{0}^{1} \int_{-\sqrt{1-x^2}}^{+\sqrt{1-x^2}} f(x,y)\,dy\,dx$$

Question 3 (15 pts) Evaluate

$$\int_{-1}^{2} \int_{y}^{\sqrt{y+2}} e^{2x^3-3x^2-12x}dxdy + \int_{-2}^{-1} \int_{-\sqrt{y+2}}^{\sqrt{y+2}} e^{2x^3-3x^2-12x}dxdy$$

(**Hint:** First write the sum of double integrals as a single double integral in $dydx$ order.)



$x = \sqrt{y+2} \iff x^2 = y+2$ and $x \geq 0$

$\iff y = x^2 - 2$ and $x \geq 0$

$$\int_{-1}^{2} \int_{y}^{\sqrt{y+2}} f(x,y)\,dx\,dy + \int_{-2}^{-1} \int_{-\sqrt{y+2}}^{+\sqrt{y+2}} f(x,y)\,dx\,dy = \int_{-1}^{2} \int_{x^2-2}^{x} e^{2x^3-3x^2-12x}\,dy\,dx = \int_{-1}^{2} y\, e^{2x^3-3x^2-12x} \Big|_{y=x^2-2}^{y=x} dx$$

$$\int_{-1}^{2} (x - x^2 + 2)\, e^{2x^3-3x^2-12x}\,dx = -\frac{1}{6}\int_{7}^{-20} e^{u}\,du = \frac{1}{6} e^{u}\Big|_{-20}^{7} = \frac{1}{6}\left(e^{7} - \frac{1}{e^{20}}\right)$$

$u = 2x^3 - 3x^2 - 12x$

$du = (6x^2 - 6x - 12)\,dx$

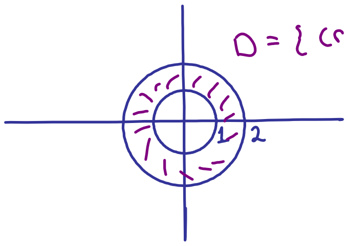
$\frac{1}{6} du = (x^2 - x - 2)\,dx$

$x = -1 \Rightarrow u = -2 - 3 + 12 = 7$

$x = 2 \Rightarrow u = 16 - 12 - 24 = -20$

Question 4 (15 pts) Let D be the region $D = \{(x, y); 1 \le x^2 + y^2 \le 4\}$. Evaluate

$$\int_D \int \ln(x^2 + y^2) dA$$



$D = \{(r, \theta) : r \in [1, 2], \theta \in [0, 2\pi]\}$

$$\iint_D \ln(x^2+y^2)\,dA = \int_0^{2\pi}\int_1^2 \ln(r^2)\cdot r\,dr\,d\theta = 2\pi \int_1^2 r\cdot \ln(r^2)\,dr$$

$x = r^2$
$dx = 2r\,dr$

$$= 2\pi \frac{1}{2}\int_1^4 \underbrace{\ln x}_{u}\,\underbrace{dx}_{dv} = \pi\left[x\ln x - \int dx\right]_1^4 = \pi\, x(\ln x - 1)\Big|_1^4$$

$du = \frac{1}{x}dx$
$v = x$

$$= \pi\left[(4\ln 4 - 4) - (-1)\right]$$

$$= \pi(4\ln 4 - 3)$$

Question 5 (15 pts) Find and classify all critical points of the function

$$f(x, y) = x^2 + y^2 - 3xy + \frac{5}{4}x^4.$$

$P_0(x_0, y_0)$ is a critical point iff $\nabla f(P_0) = \langle f_x(P_0), f_y(P_0)\rangle = \vec{0} = \langle 0, 0\rangle$

$$\left.\begin{aligned} f_x(x,y) &= 2x - 3y + 5x^3 \overset{(1)}{=} 0 \\ f_y(x,y) &= 2y - 3x \underset{(2)}{=} 0 \end{aligned}\right\} \quad (2) \Rightarrow y = \frac{3}{2}x \overset{(1)}{\Rightarrow} 2x - 3\left(\frac{3}{2}x\right) + 5x^3 = 0$$

$$\Rightarrow x\left(2 - \frac{9}{2} + 5x^2\right) = 0$$

$$\Rightarrow x\left(5x^2 - \frac{5}{2}\right) = 0$$

$$\Rightarrow x_1 = 0, \quad x_2 = \frac{1}{\sqrt{2}}, \quad x_3 = \frac{-1}{\sqrt{2}}$$

The set of all critical points is $\{P_1, P_2, P_3\}$ where

$$P_1 = (0,0), \quad P_2\left(\frac{1}{\sqrt{2}}, \frac{3}{2}\frac{1}{\sqrt{2}}\right), \quad P_3\left(\frac{-1}{\sqrt{2}}, \frac{-3}{2}\frac{1}{\sqrt{2}}\right)$$

$$\left.\begin{aligned} f_{xx} &= 2 + 15x^2 \\ f_{yy} &= 2 \\ f_{xy} &= -3 \end{aligned}\right\} \quad \Delta(x,y) = f_{xx}(x,y)\cdot f_{yy}(x,y) - f_{xy}^2(x,y) = 2(2 + 15x^2) - 9$$

Hence $\Delta(P_1) = -5 < 0 \Rightarrow P_1$ is a saddle point (neither local max, nor local min)

and $\left[\begin{aligned} \Delta(P_2) &= 2\left(2 + 15\tfrac{1}{2}\right) - 9 > 0 \\ f_{xx}(P_2) &= 2 + 15\tfrac{1}{2} > 0 \end{aligned}\right\}$ P_2 is a local minimum point.

$\left.\begin{aligned} &\Delta(P_3) = \Delta(P_2) > 0 \\ &\text{and} \\ &f_{xx}(P_3) = f_{xx}(P_2) > 0 \end{aligned}\right\}$ P_3 is also a local minimum point

Question 6 (8+4=12 pts) Let $f(x, y)$ be a function which has continuous partial derivatives at the point P_0. For the unit vectors $\vec{u} = \frac{1}{\sqrt{2}}\langle 1, 1\rangle$, and $\vec{v} = \frac{1}{\sqrt{5}}\langle 2, -1\rangle$, directional derivatives of f are given as $D_{\vec{u}}f(P_0) = -\sqrt{2}$ and $D_{\vec{v}}f(P_0) = \sqrt{5}$.

a) Find the directional derivative of f at the point P_0 in the direction of the vector $\langle 1, 2\rangle$.

Since f has continuous partial derivatives at P_0, it is differentiable at P_0 and hence for any unit vector $\vec{a} = \langle a_1, a_2\rangle$, $D_{\vec{a}} f(P_0) = \nabla f(P_0) \cdot \vec{a} = f_x(P_0) \cdot a_1 + f_y(P_0)\, a_2$

Hence at P_0 we have $\begin{bmatrix} f_x \frac{1}{\sqrt{2}} + f_y \frac{1}{\sqrt{2}} = -\sqrt{2} \\ f_x \frac{2}{\sqrt{5}} + f_y\left(\frac{-1}{\sqrt{5}}\right) = \sqrt{5} \end{bmatrix} \iff \begin{bmatrix} f_x + f_y = -2 \\ 2f_x - f_y = 5 \end{bmatrix}$

Thus $3 f_x = 3 \Rightarrow f_x(P_0) = 1$

and $f_y = -2 - f_x \Rightarrow f_y(P_0) = -2 - 1 \Rightarrow f_y(P_0) = -3$

$\vec{a} = \frac{1}{\sqrt{5}}\langle 1, 2\rangle$ is the unit vector in direction of $\langle 1, 2\rangle$, so $D_{\frac{1}{\sqrt{5}}\langle 1,2\rangle} f(P_0) = \underset{\langle 1,-3\rangle}{\nabla f(P_0)} \cdot \langle 1, 2\rangle \frac{1}{\sqrt{5}} = \frac{1}{\sqrt{5}}(1-6) = -\sqrt{5}$

b) Find the direction $\vec{w}$ in which f increases the most rapidly at P_0.

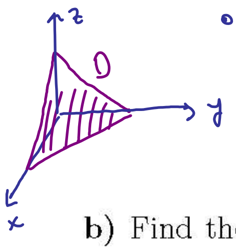
Since $D_{\vec{w}} f(P_0) = \nabla f(P_0) \cdot \vec{w} = \|\nabla f(P_0)\| \cdot \cos\theta \le \|\nabla f(P_0)\|$ and it attains it's maximum value if $\theta = 0$, that is if $\vec{w} = \frac{\nabla f(P_0)}{\|\nabla f(P_0)\|}$

Thus $\vec{w} = \frac{\langle 1, -3\rangle}{\sqrt{1+9}} = \frac{1}{\sqrt{10}}\langle 1, -3\rangle$ (You may also take $\vec{w} = \nabla f(P_0) = \langle 1, -3\rangle$ or any vector in this direction)

Question 7 (4+12=16 pts)

a) Explain why there are maximum and minimum values of the function $f(x, y, z) = xyz^2$ subject to $x + y + z = 1$, $x \ge 0$, $y \ge 0$, and $z \ge 0$.

Since $f(x,y,z)$ is a polynomial in 3 variables, it is continuous everywhere, in particular on the closed and bounded region D, and hence f has max/min on D.



b) Find the maximum and the minimum stated in part (a) using Lagrange Multipliers method.

Find Max/min of $f(x,y,z) = xyz^2$ subject to $g(x,y,z) = x+y+z-1 = 0$ } set $\nabla f = \lambda \nabla g$

$$\left.\begin{aligned} yz^2 &\overset{1}{=} \lambda \\ xz^2 &\overset{2}{=} \lambda \\ 2xyz &\overset{3}{=} \lambda \\ x+y+z &\overset{4}{=} 1 \end{aligned}\right]$$

4 equations in 4 unknowns ☺

First of all since $x \geq 0$, $y \geq 0$, $z \geq 0$ and $f(x,y,z) = xyz^2 \geq 0$ $\forall (x,y,z) \in D$ and on the boundary of D that is if $x = 0$ or if $y = 0$ or if $z = 0$, $f(x,y,z) = 0$. Hence minimum value of f on D is zero and it takes this value on the edges of triangle. Now assume $x \neq 0$, $y \neq 0$, $z \neq 0$.

(1) & (2) $\Rightarrow y\cancel{z^2} = x\cancel{z^2} \Rightarrow y = x \underset{3,4}{\Longrightarrow} \left.\begin{matrix} 2x^2 z = \lambda \\ 2x + z = 1 \end{matrix}\right\}$ $\quad 2x^{\cancel{2}}\cancel{z} = \cancel{x}z^{\cancel{2}} \Rightarrow 2x = z$; $\quad 2x + z = 1 \longrightarrow x = \frac{1}{4} = y \Rightarrow z = \frac{1}{2}$
$(z \neq 0)$ $\qquad xz^2 = \lambda$

$\Rightarrow MAX = f\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right) = \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{64}$

M E T U
Department of Mathematics

Group	CALCULUS WITH ANALYTIC GEOMETRY II Final Exam	List No.

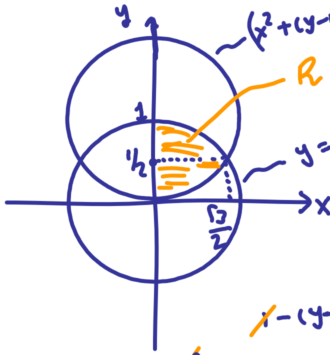
Code : *Math 120* Acad. Year : *2011-2012* Semester : *Spring* Coordinator: *Muhiddin Uğuz*	Last Name : Name : Student No. : Department : Section : Signature :
Date : *May.31.2012* Time : *9:30* Duration : *150 minutes*	9 QUESTIONS ON 6 PAGES TOTAL 100 POINTS

1	2	3	4	5	6	7	8	9		

Question 1 (3+4+3+4=14 pts) Let R be the region bounded by the curves $x = 0$, $x^2 + (y-1)^2 = 1$ and $x^2 + y^2 = 1$ containing the point $P(\frac{1}{2}, \frac{1}{2})$. Write the double integral $\int\int_R f(x,y)dA$ as an iterated integral:

a) Using cartesian coordinates in $dydx$ order.

$$\int_0^{\sqrt{3}/2} \int_{1-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x,y)\, dy\, dx$$



$\cancel{1}-(y-1)^2 = \cancel{1}-y^2 \; (=x^2)$

$-\cancel{y^2}+2y-1 = -\cancel{y^2}$

$y = 1/2 \Rightarrow x = \frac{\sqrt{3}}{2}$

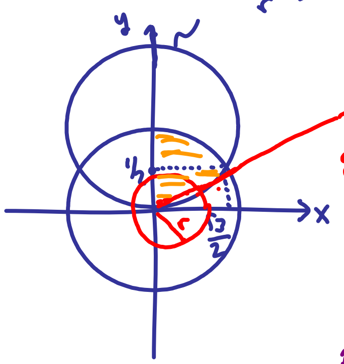
b) Using cartesian coordinates in $dxdy$ order.

$$\int_0^{1/2} \int_0^{\sqrt{1-(y-1)^2}} f(x,y)\, dx\, dy + \int_{1/2}^{1} \int_0^{\sqrt{1-y^2}} f(x,y)\, dx\, dy$$

$y^2-2y+1+x^2=1 \Rightarrow x^2+y^2=2y$

$r^2 = 2r\sin\theta \Rightarrow r = 2\sin\theta$

$\Rightarrow \theta = \arcsin(\frac{r}{2})$

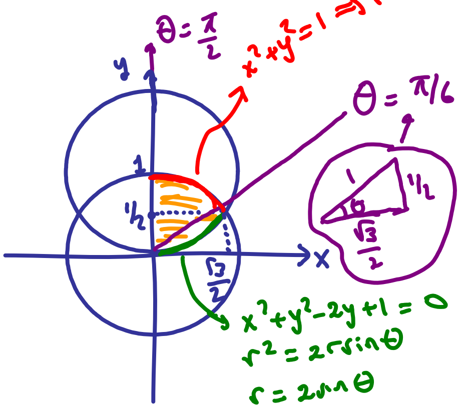


given $r \in [0,1]$, θ changes from $\arcsin\frac{r}{2}$ to $\pi/2$

c) Using polar coordinates in $d\theta dr$ order.

$$\int_0^1 \int_{\arcsin\frac{r}{2}}^{\pi/2} f(r\cos\theta, r\sin\theta)\, r\, d\theta\, dr$$

c) Using polar coordinates in $drd\theta$ order.



$x^2+y^2=1 \Rightarrow r=1$

$x^2+y^2-2y+1=0$

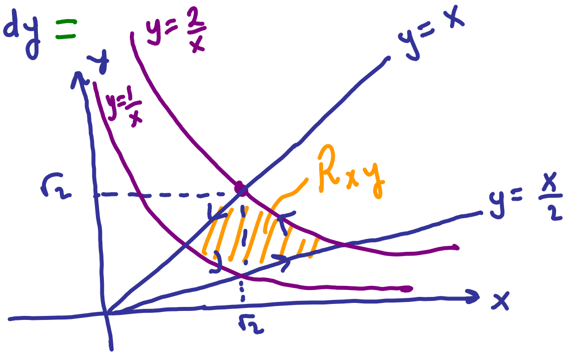
$r^2 = 2r\sin\theta$

$r = 2\sin\theta$

$$\int_0^{\pi/6} \int_0^{2\sin\theta} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta + \int_{\pi/6}^{\pi/2} \int_0^{1} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Question 2 (12 pts) Evaluate $\oint_C \frac{y^2}{x}\,dx + y\ln x\,dy$ where C is the counterclockwisely oriented boundary of the region in the first quadrant (i.e., in the region where $x \geq 0$, and $y \geq 0$) bounded by the curves $y = x$, $y = \frac{x}{2}$, $y = \frac{1}{x}$, and $y = \frac{2}{x}$.

$$\oint_C \frac{y^2}{x^2}dx + y\ln x\,dy = \oint_C M(x,y)\,dx + N(x,y)\,dy =$$



$$\left.\begin{array}{l} M_y(x,y) = \frac{2y}{x} \\ N_x(x,y) = \frac{y}{x} \end{array}\right\} \Rightarrow N_x - M_y = \frac{-y}{x}$$

Since components of $\vec{F} = \langle M, N\rangle$ have continuous partial derivatives on R_{xy} and $\partial R_{xy} = C$ is positively oriented closed curve, we can use Green's Thm:

$$= \iint_{R_{xy}} \frac{-y}{x}\,dA = \iint_{R_{xy}} f(x,y)\,dy\,dx = \iint_{R_{uv}} \underbrace{f(x(u,v), y(u,v))}_{= -u} \underbrace{\left|\frac{\partial(x,y)}{\partial(u,v)}\right|}_{= \frac{1}{\left|\frac{\partial(u,v)}{\partial(x,y)}\right|} = \frac{1}{|-2u|} = \frac{1}{2u}} du\,dv$$

$$\boxed{\begin{array}{l} \text{Let } \left.\begin{array}{l} u = \frac{y}{x} \\ v = xy \end{array}\right\} \begin{array}{l} y = \frac{1}{x} \leftrightarrow v = 1, \quad y = x \leftrightarrow u = 1 \\ y = \frac{2}{x} \leftrightarrow v = 2, \quad y = \frac{x}{2} \leftrightarrow u = \frac{1}{2} \end{array} \\ \frac{\partial(u,v)}{\partial(x,y)} = \det\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \det\begin{bmatrix} \frac{-y}{x^2} & \frac{1}{x} \\ y & x \end{bmatrix} = -\frac{y}{x} - \frac{y}{x} = \frac{-2y}{x} = -2u \end{array}}$$

$$= \iint_{R_{uv}} -u \, \frac{1}{2u}\,du\,dv = \frac{-1}{2}\int_1^2 \int_{1/2}^1 1\,du\,dv = \frac{-1}{2}\text{Area}(R_{uv})$$

$$= \frac{-1}{2}\cdot\frac{1}{2} = \frac{-1}{4}$$

Question 3 (8+4=12 pts) Given the power series $\sum_{n=1}^{\infty}(-1)^n \frac{(x-1)^{2n}}{n}$.

a) Find its radius of convergence R, and its interval of convergence I.

Let's apply ratio test: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{|x-1|^{2n+2}}{n+1}\cdot\frac{n}{|x-1|^{2n}} = |x-1|^2 \lim_{n\to\infty}\frac{n}{n+1}$

$= |x-1|^2$

Thus given power series is convergent if $|x-1|^2 < 1$, that is if $|x-1| < 1$. Hence $R = 1$

0 1 2

To determine interval of convergence I, lets check the end points:

$x = 0 \Rightarrow \sum_{n=1}^{\infty}(-1)^n \frac{(-1)^{2n}}{n} = \sum (-1)^n \left(\frac{1}{n}\right) = b_n$ $(b_n > 0, \lim_{n\to\infty} b_n = 0, b_n \searrow) \Rightarrow \sum(-1)^n b_n$ is convergent by Alternating series test.

$x = 2 \Rightarrow \sum_{n=1}^{\infty}(-1)^n \frac{1^n}{n} = \sum_{n=1}^{\infty}(-1)^n \frac{1}{n}$ convergent by AST.

Hence $I = [0, 2]$

b) Find the value of the infinite sum in part (a) when $x = \frac{1}{2}$ approximately within $|\text{error}| \le \frac{1}{120}$.

$x = \frac{1}{2} \in [0, 2] \Rightarrow \sum_{n=1}^{\infty}(-1)^n \frac{\left(\frac{1}{2}-1\right)^{2n}}{n} = \sum_{n=1}^{\infty}(-1)^n \frac{\left(-\frac{1}{2}\right)^{2n}}{n}$

$= \sum_{n=1}^{\infty}(-1)^n \frac{1}{2^{2n}\cdot n}$ is an alternating series and $\frac{1}{4^n \cdot n} \searrow 0$, hence by Alt. series test $|S_N - S| < |a_{N+1}|$

$= -\frac{1}{4} + \frac{1}{32} - \frac{1}{64\cdot 3} + \cdots$

OR

We want $|a_{N+1}| < \frac{1}{120}$ ie $\frac{1}{4^n\cdot n} < \frac{1}{120}$. This is true $\forall n \geq 3$. Hence take $S \approx S_2$

$\approx -\frac{1}{4} + \frac{1}{32} = S_2$

$|\text{error}| < \left|\frac{-1}{64\cdot 3}\right| < \frac{1}{120}$

Question 4 (6 pts) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence satisfying $\lim_{n\to\infty} n^2 a_n = \frac{-1}{2012}$. Determine whether $\sum_{n=1}^{\infty} a_n$ is convergent or divergent.

Consider $\sum |a_n|$. Since $\lim_{n\to\infty}\frac{|a_n|}{\frac{1}{n^2}} = \lim_{n\to\infty} n^2|a_n| = \left|\frac{-1}{2012}\right|$ a non zero finite number, by limit comparison test, either both $\sum |a_n|$ and $\sum \frac{1}{n^2}$ converge or both diverge. Since $\sum \frac{1}{n^2}$ is convergent by p-test (or by integral test) so is $\sum |a_n|$. Thus $\sum a_n$ is (absolutely) convergent.

Question 5 (8 pts) Given that $f'(u) = \sin(u^2)$, $f(1) = 1$, and $z(x, y) = f(xf(y^2))$, evaluate $\frac{\partial^2 z}{\partial x \partial y}$ when $(x, y) = (1, 1)$.

$f''(u) = \cos(u^2) \cdot 2u$

$$\frac{\partial^2 z}{\partial x \partial y} = z_{yx} = (z_y)_x = \frac{\partial}{\partial x} z_y = \frac{\partial}{\partial x}\left[\underline{f'(x f(y^2))} \cdot x\, f'(y^2) \cdot 2y \right]$$

$$= 2y \cdot f'(y^2) \cdot \left[f''(x f(y^2)) \cdot f(y^2) \cdot x + f'(x f(y^2)) \right]$$

$$= 2y \cdot \sin(y^4) \left[2\, x f(y^2) \cdot \cos\left(x^2 f^2(y^2)\right) \cdot f(y^2) \cdot x + \sin\left(x^2 f^2(y^2)\right) \right]$$

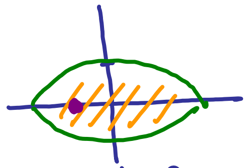
$x=1$, $y=1$:

$$= 2\sin(1) \left[2 \overbrace{f(1)}^{1} \cos\left(\overbrace{f^2(1)}^{1}\right) f(1) + \sin(f^2(1)) \right]$$

$$= 2\sin(1)\left[2\cos(1) + \sin(1)\right]$$

Question 6 (12 pts) Find the absolute maximum and the absolute minimum of $z = f(x, y) = x^2 + x - y^2$ on $D = \{(x, y);\ x^2 + 2y^2 \le 1\}$.

First notice that $f(x,y)$ is a polynomial in two variables, and hence continuous on the closed & bounded domain D. Therefore f has absolute MAX/min on D. Since f has no singular points (ie pts at which f_x or f_y does not exist), abs. max/min of f are either at critical pts or on the boundary of D.



- Lets first find all critical points inside D:

$$\left.\begin{array}{l} f_x = 2x+1 \\ f_y = -2y \end{array}\right\} \quad f_x = 0 = f_y \iff \boxed{(x,y) = \left(-\tfrac{1}{2}, 0\right)}$$

is the only critical point, and it lies inside D.

- Secondly check the boundary pts of D: i.e.: (x,y) s.t. $x^2 + 2y^2 = 1$

ie/ find Max/min of $f(x,y) = x^2 + y - y^2$ subject to $g(x,y) = x^2 - 2y^2 - 1 = 0$ — use Lagrange Multiples method

set $\nabla f = \lambda \nabla g$:

$$\boxed{\begin{array}{ll} f_x = 2x+1 \overset{1}{=} \lambda(2x) = \lambda g_x \\ f_y = -2y \overset{2}{=} \lambda(4y) = \lambda g_y \\ x^2 + 2y^2 \overset{3}{=} 1 \end{array}}$$

– 3 unknowns (x, y, λ), 3 equations ☺

$y = 0 \overset{(3)}{\Rightarrow} x = \mp 1 \Rightarrow (+1, 0), (-1, 0)$

$y \neq 0 \overset{(2)}{\Rightarrow} -1 = 2\lambda \Rightarrow \lambda = -\frac{1}{2} \overset{(1)}{\Rightarrow} 2x+1 = -x \Rightarrow x = -\frac{1}{3} \overset{(3)}{\Rightarrow} y = \mp\frac{2}{3} \Rightarrow (x,y) = \left(-\frac{1}{3}, \mp\frac{2}{3}\right)$

OR On the boundary: $x^2 + 2y^2 = 1 \Rightarrow y^2 = \frac{1}{2} - \frac{1}{2}x^2$

$\Rightarrow$ on the boundary points $f(x,y) = x^2 - x - \left(\frac{1}{2} - \frac{1}{2}x^2\right)$

$\Rightarrow$ Equivalently, find Max/min of $\varphi(x) = \frac{3}{2}x^2 + x - \frac{1}{2}$ on $[-1, +1]$: $\varphi'(x) = 3x + 1 = 0 \Rightarrow x = -1/3$

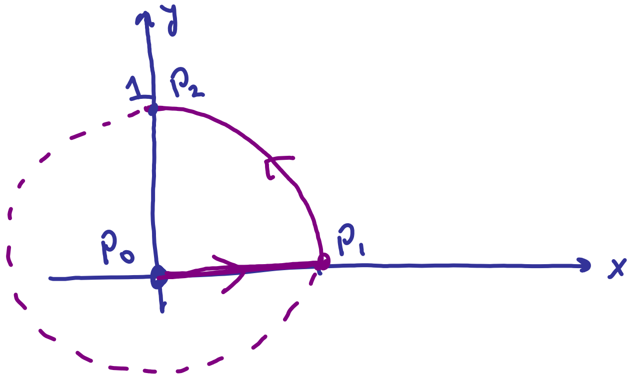
$\Rightarrow \varphi(-1/3) = f(-1/3, \mp\frac{2}{3}) = -\frac{2}{3}$, $\varphi(-1) = f(-1, 0) = 0$

$\varphi(+1) = f(+1, 0) = 2$; $f(-\frac{1}{2}, 0) = -1/4$

- $f(-\frac{1}{2}, 0) = -1/4$
  $f(+1, 0) = 2 \longrightarrow$ MAX
  $f(-1, 0) = 0$
  $f(-\frac{1}{3}, \mp\frac{2}{3}) = -\frac{2}{3} \longrightarrow$ min

Surname: .... Name: .... Student id: .... Signature: ....

Question 7 (3+4+3+4=14 pts) Consider the curve C that starts from the point $P_0 = (0,0)$ goes to the point $P_1 = (1,0)$ along the line segment and then from the point $P_1 = (1,0)$ to the point $P_2 = (0,1)$ along the part of the circle $x^2 + y^2 = 1$ and let $\vec{F}$ be the vector field $\vec{F}(x,y) = \langle e^x + y, y^2 + x \rangle$.



a) Prove that $\vec{F}$ is conservative in R^2.

$\vec{F} = \langle P, Q \rangle$ where $P(x,y) = e^x + y$

$Q(x,y) = y^2 + x$

Since $P_y = 1 = Q_x$ and P, Q have continuous partial derivatives on $\mathbb{R}^2$, $\vec{F}$ is conservative.

b) Find a potential function $\phi(x,y)$ for $\vec{F}$.

$\nabla \phi(x,y) = \langle \phi_x, \phi_y \rangle = \vec{F} = \langle P, Q \rangle \Rightarrow$

$\phi_x = P = e^x + y \Rightarrow \phi(x,y) = e^x + xy + c(y) \Rightarrow \phi_y = x + c'(y)$

$\phi_y = Q \Rightarrow x + c'(y) = x + y^2 \Rightarrow c'(y) = y^2 \Rightarrow c(y) = \frac{y^3}{3} + c$

$\Rightarrow \phi(x,y) = e^x + xy + \frac{y^3}{3}$ (for $c = 0$)

c) Evaluate $\int_C \vec{F} \cdot d\vec{r}$ using the potential function you obtained in part (b).

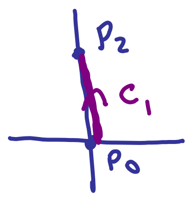
$$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla \phi \cdot d\vec{r} = \phi(\text{terminal pt.}) - \phi(\text{initial pt})$$

$$= \phi(0,1) - \phi(0,0) = (e^0 + 0 + \tfrac{1}{3}) - (e^0 + 0 + 0)$$

$$= 1 + \tfrac{1}{3} - 1 = 1/3$$

d) Choose a suitable path and evaluate $\int_C \vec{F} \cdot d\vec{r}$ using path independence.

We can choose the line segment from $P_0\,(0,0)$ to $P_2\,(0,1)$



$$\int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_1} P\,dx + Q\,dy = \int_0^1 [(e^0 + t) \cdot 0 + (t^2 + 0)\, 1]\, dt$$

$$= \int_0^1 t^2\,dt$$

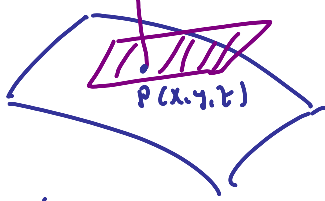
$$= \frac{t^3}{3}\Big|_0^1 = \frac{1}{3}$$

$r(t) = (0, t) \Rightarrow dx = \frac{dx}{dt}\,dt = 0\,dt = 0$

$t \in [0,1]$ $\quad dy = \frac{dy}{dt}\,dt = 1\,dt$

Question 8 (10 pts) Find all points on the surface $xyz + yz + xz^2 = -1$ at which tangent planes are perpendicular to the vector $\vec{i} - 2\vec{k}$.

$\vec{n} \parallel \nabla F(P) = \langle yz + z^2,\ xz + z,\ xy + y + 2xz \rangle$



$F(x,y,z) = xyz + yz + xz^2 + 1 = 0$

we have

1) $P(x,y,z)$ is on the surface : $xyz + yz + xz^2 = -1$

2) $\nabla F(P) \parallel \langle 1, 0, -2 \rangle \Rightarrow \nabla F(P) = \lambda \langle 1, 0, -2 \rangle \quad (\lambda \neq 0)$

1) $yz + z^2 = \lambda$

2) $xz + z = 0$

3) $xy + y + 2xz = -2\lambda$

4) $xyz + yz + xz^2 + 1 = 0 \Rightarrow z \neq 0$

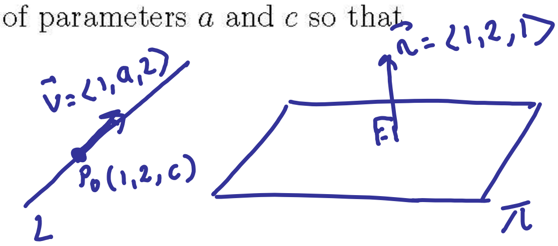
$\Rightarrow \boxed{x = -1} \xrightarrow{(3)} -2z = -2\lambda \Rightarrow \lambda = z^{*\,0} \xrightarrow{(1)} y + z \overset{(5)}{=} 1 \Downarrow (1-y)^2 = 1$

$\boxed{x = -1} \xrightarrow{(4)} -yz + yz - z^2 + 1 = 0 \Rightarrow z^2 = 1 \Rightarrow (1-y)^2 = 1$

$y = 0 \overset{(5)}{\Downarrow} z = 1 \qquad y = 2 \overset{(5)}{\Downarrow} z = -1$

$\therefore (x,y,z) \in \{(-1,0,1),\ (-1,2,-1)\}$

Question 9 (4+4+4=12 pts) Given the plane $\Pi : x + 2y + z = 1$ and the line $L : x - 1 = \dfrac{y-2}{a} = \dfrac{z-c}{2}$, determine all values of parameters a and c so that



a) L is parallel to Π but L does not intersect Π.

$L \parallel \pi \iff \vec{v} \cdot \vec{n} = 0$

$0 = \langle 1, a, 2 \rangle \cdot \langle 1, 2, 1 \rangle = 1 + 2a + 2$

$\boxed{a = \frac{-3}{2}}$

and $P_0 \notin \pi \iff \cancel{1} + 4 + c \neq \cancel{1} \iff \boxed{c \neq -4}$

b) L lies on Π.

L lies on $\pi \iff \left[\left(L \parallel \pi \iff \boxed{a = -3/2} \right) \text{ and } \left(P_0 \in \pi \right) \right]$

$\iff \boxed{a = -3/2 \quad \text{and} \quad c = -4}$

c) L intersects with Π only at one point.

$\iff L \not\parallel \pi \iff \boxed{a \neq \frac{-3}{2} \ \ \& \ \ c \in \mathbb{R}}$