M E T U
Department of Mathematics

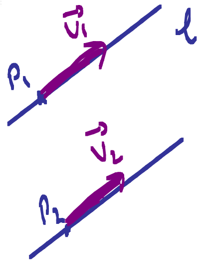
Group	CALCULUS WITH ANALYTIC GEOMETRY II MidTerm 1	List No.

Code : *Math 120* Acad. Year : *2012-2013* Semester : *Spring* Coordinator: *Muhiddin Uğuz* Date : *April.06.2013* Time : *9:30* Duration : *120 minutes*	Last Name : Name : Student No. : Department : Section : Signature : 8 QUESTIONS ON 4 PAGES TOTAL 100 POINTS

1	2	3	4	5	6	7	8	SHOW YOUR WORK

Question 1 (12 pts) Consider the line $l_1 : \frac{x+1}{2} = \frac{y-3}{1}, z = 1$ and the line $l_2 : x = -1 + 2t, y = 3 + t, z = 1 - \sqrt{5}t$.

a) Show that the lines l_1 and l_2 intersect.



l_1 : $P_1(-1,3,1)$, $\vec{v}_1 = \langle 2,1,0\rangle$

l_2 $P_2(-1,3,1)$, $\vec{v}_2 = \langle 2,1,-\sqrt{5}\rangle$

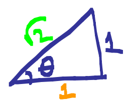
since for each point on l_1, $z=1$,
we must have $1 = 1-\sqrt{5}t \Rightarrow t=0$
Hence if $l_1 \cap l_2 \neq \emptyset$, they must intersect when $t=0$. so the only possible intersection point is $P(-1,3,1) \in l_2$
since P is also on l_1 ($x=-1 \Rightarrow \frac{x+1}{2} = 0 = \frac{y-3}{1} \Rightarrow y=3$ ✓)
These two lines intersects : $l_1 \cap l_2 = P(-1,3,1)$

b) Find the acute angle $(0 \leq \theta \leq \pi/2)$ between the lines l_1 and l_2.

The acute angle θ between lines l_1 & l_2 is the acute angle between their direction vectors $\vec{v}_1$ & $\vec{v}_2$. since $\vec{v}_1 \cdot \vec{v}_2 = \|\vec{v}_1\| \, \|\vec{v}_2\| \cos\theta$

and $\left.\begin{array}{l} \vec{v}_1 \cdot \vec{v}_2 = \langle 2,1,0\rangle \cdot \langle 2,1,-\sqrt{5}\rangle = 4+1 = 5 \\ \|\vec{v}_1\| = \sqrt{5} \\ \|\vec{v}_2\| = \sqrt{10} = \sqrt{2}\cdot\sqrt{5} \end{array}\right\} \cos\theta = \frac{5}{\sqrt{5}\,\sqrt{2}\cdot\sqrt{5}} = \frac{1}{\sqrt{2}}$

Hence $\theta = \arccos\left(\frac{1}{\sqrt{2}}\right) = \pi/4$



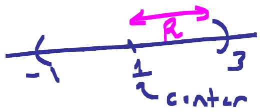
Question 2 (16 pts)

Find the radius and of interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(x-1)^{3n}}{n8^n}$.

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{|x-1|^{3n+3}}{(n+1)\,8^{n+1}} \cdot \frac{n\,8^n}{|x-1|^{3n}} = \frac{|x-1|^3}{8} \cdot \lim_{n\to\infty} \frac{n}{n+1} = \frac{|x-1|^3}{8}$

Hence given power series is (absolutely) convergent if $\frac{|x-1|^3}{8} < 1$

$\Leftrightarrow |x-1|^3 < 8 \Leftrightarrow |x-1| < 2$
$-2 < x-1 < 2$
$-1 < x < 3$



Radius of convergence is $R = 2$

Lets check the end points;

$x=-1 \Rightarrow \sum \frac{(-2)^{3n}}{n8^n} = \sum \frac{(-1)^{3n}\,\cancel{(2^3)^n}}{n\cdot \cancel{8^n}} = \sum \frac{(-1)^n}{n}$ convergent alternating series

$x=3 \Rightarrow \sum \frac{2^{3n}}{n8^n} = \sum \frac{1}{n}$ divergent harmonic series (or divergent by p-test)

Thus $I = [-1,3)$

Question 3 (12 pts) Determine whether the following series are convergent or divergent.

a) $\displaystyle\sum_{n=1}^{\infty} \cos\left(\frac{n^2}{1+n^4}\right)$

$\cos x$ is continuous

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \cos\left(\frac{n^2}{1+n^4}\right) = \cos\left(\lim_{n\to\infty} \frac{n^2}{1+n^4}\right) = \cos 0 = 1 \neq 0$$

Hence by n^{th} term test, $\sum a_n$ is divergent.

b) $\displaystyle\sum_{n=1}^{\infty} \frac{1+e^{-n}}{\sqrt[3]{n^2+120}} = \sum_{n=1}^{\infty} a_n$, $a_n \geq 0$, let $b_n = \frac{1}{n^{2/3}} > 0 \quad \forall n \geq 1$.

$$\lim_{n\to\infty} \frac{a_n}{\frac{1}{n^{2/3}}} = \lim_{n\to\infty} \frac{1+e^{-n}}{\sqrt[3]{n^2+120}} \cdot n^{2/3} = \lim_{n\to\infty} \underbrace{\frac{n^{2/3}}{\sqrt[3]{n^2+120}}}_{\to 1} \cdot \underbrace{\left(1+\frac{1}{e^n}\right)}_{\to 1} = 1$$

since limit is a nonzero number, we have $\sum a_n$ is convergent if and only if $\sum \frac{1}{n^{2/3}}$ is convergent.

On the other hand, since $\sum \frac{1}{n^{2/3}}$ is divergent by p-test ($p = \frac{2}{3} < 1$), we conclude that $\sum a_n$ is divergent.

Question 4 (12 pts) Determine whether the following series are convergent or divergent.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{2^{2n+1}} = \sum a_n$; since $a_n \geq 0$ we can apply Ratio Test:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)\cancel{n!}}{2^2 \cdot \cancel{2^{2n+1}}} \cdot \frac{\cancel{2^{2n+1}}}{\cancel{n!}} = \frac{1}{4} \lim_{n\to\infty} (n+1) = +\infty > 1$$

Hence by Ratio Test $\sum a_n$ is divergent.

b) $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^2}\right) \sin^3 n = \sum a_n$

consider $\sum |a_n|$.

$$0 \leq \lim_{n\to\infty} \frac{|a_n|}{\left(\frac{1}{n^2}\right)} = \lim_{n\to\infty} \left|\frac{\sin\frac{1}{n^2}}{1/n^2}\right| \cdot \overbrace{|\sin^3 n|}^{\leq 1} \leq \lim_{x\to\infty} \frac{|\sin 1/x^2|}{1/x^2} = \lim_{x\to 0} \frac{|\sin x|}{x} = 1$$

(or: $|a_n| \leq \sin\left(\frac{1}{n^2}\right)$ and $\lim_{n\to\infty} \frac{\sin(1/n^2)}{1/n^2} = 1$, by limit comp. test, we have $\sum \sin\frac{1}{n^2}$ is convergent as $\sum \frac{1}{n^2}$ is conv. by p test. Hence $\sum |a_n|$ is conv. by comparison test, Thus $\sum a_n$ is abs. conv.)

Thus by Limit Comparison test $\sum |a_n|$ is convergent if and only if $\sum \frac{1}{n^2}$

since by p-test ($p = 2 > 1$), $\sum \frac{1}{n^2}$ is convergent, so is $\sum |a_n|$

Hence $\sum a_n$ is (absolutely) convergent.

or/ since $|\sin x| \leq x \quad \forall x > 0$ we have $|\sin \frac{1}{n^2}| \leq \frac{1}{n^2}$

so $0 \leq \sum |a_n| \leq \underbrace{\sum \frac{1}{n^2}}_{\text{conv. by p test}} \Longrightarrow \sum a_n$ abs. conv.

Question 5 (14 pts)

a) Is the series $\sum_{n=2}^{\infty} \frac{\cos(n\pi)}{\sqrt{n}}$ divergent, conditionally convergent or absolutely convergent?

$\cos(n\pi) = (-1)^n$, $\left| \frac{\cos(n\pi)}{\sqrt{n}} \right| = \frac{1}{\sqrt{n}}$, $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$ is divergent by p-test ($p = \frac{1}{2} < 1$). Hence $\sum a_n$ is not absolutely convergent.

Since $a_n = \frac{(-1)^n}{\sqrt{n}}$, $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$. Moreover $\frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}$ (since $n+1 > n > 1$ and $\sqrt{n+1} > \sqrt{n} > 0$). So by Alternating Series Test, $\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ is convergent.

Thus $\sum \frac{(-1)^n}{\sqrt{n}}$ is conditionally convergent

b) Is the series $\sum_{n=0}^{\infty} \ln\left(\frac{2n+5}{2n+3}\right)$ convergent or divergent? If convergent, find the value of the sum of the series.

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} [\ln(2n+5) - \ln(2n+3)] = \lim_{n\to\infty} S_n \overset{(*)}{=} \lim_{n\to\infty} \ln\left(\frac{2n+3}{3}\right) = \infty$$

Divergent.

$$(*): \boxed{\begin{aligned} S_n &= a_0 + a_1 + \cdots + a_{n-1} \\ &= (\cancel{\ln 5} - \ln 3) + (\cancel{\ln 7} - \cancel{\ln 5}) + (\ln 9 - \cancel{\ln 7}) + \cdots + (\ln(2n+3) - \cancel{\ln(2n+1)}) \\ &= \ln(2n+3) - \ln 3 = \ln\left(\frac{2n+3}{3}\right) \end{aligned}}$$

Question 6 (8 pts) Is the sequence $\left\{ \frac{(-1)^n n^3}{2^n} \right\}_{n=1}^{\infty}$ convergent or divergent? Explain your answer.

$\left| \frac{(-1)^n n^3}{2^n} \right| = \frac{n^3}{2^n}$, $\lim_{x\to\infty} \frac{x^3}{2^x}$ $[\infty/\infty]$ $\overset{L'H}{=} \lim_{x\to\infty} \frac{3x^2}{(\ln 2)\, 2^x}$ $[\infty/\infty]$ $\overset{L'H}{=} \lim_{x\to\infty} \frac{6x}{\ln^2 2 \cdot 2^x}$ $[\infty/\infty]$ $\overset{L'H}{=} \lim_{x\to\infty} \frac{6}{\ln^3 2 \cdot 2^x} = 0$. So $\lim_{n\to\infty} \frac{n^3}{2^n} = 0$ and hence $\lim_{n\to\infty} \frac{(-1)^n \cdot n^3}{2^n} = 0$

OR Consider $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^n \cdot n^3}{2^n}$

$\left| \frac{a_{n+1}}{a_n} \right| = \frac{(n+1)^3}{2 \cdot \cancel{2^n}} \cdot \frac{\cancel{2^n}}{n^3} = \frac{1}{2}\left(\frac{n+1}{n}\right)^3 \xrightarrow{n\to\infty} \frac{1}{2} < 1$ $\Rightarrow$ $\sum |a_n|$ is conv. by Ratio Test

$\Rightarrow \lim_{n\to\infty} |a_n| = 0$ by n^{th} term test.

$\Rightarrow \lim_{n\to\infty} a_n = 0$.

Question 7 (14 pts)

Given $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ for all x, let $F(x) = \int_0^x t\cos(t^3)\, dt$.

a) Find Maclaurin series of $F(x)$.

$\cos t^3 = \sum_{n=0}^{\infty} \frac{(-1)^n t^{6n}}{(2n)!}$, $t.\cos t^3 = \sum_{n=0}^{\infty} \frac{(-1)^n t^{6n+1}}{(2n)!}$ $\forall t$.

$\int_0^x t\cos(t^3)\,dt = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+2}}{(6n+2)(2n)!}$ $\forall x$

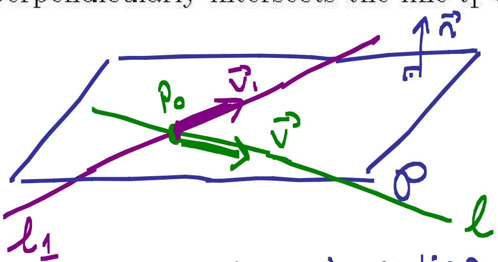
b) Find an approximation to $F(\frac{1}{2})$ with $|\text{error}| < 10^{-4}$

$F(\frac{1}{2}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{6n+2}(6n+2)(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{8.64^n(3n+1)(2n)!}$. This is an alternating series where $|a_n| = \frac{1}{8.64^n.(3n+1)(2n)!}$ is clearly decreasing and limit is 0.

So $|s-s_n| < \frac{1}{8.64^{n+1}(3n+4)(2n+2)!} < \frac{1}{10.000}$

$n=1 \Rightarrow 8.64^2.7.4! > 10^4$. Hence $F(\frac{1}{2}) \approx S_1 = \frac{1}{8} - \frac{1}{4096}$

Question 8 (12 pts) Find an equation (in any form) of the **line** l which lies on the plane $\mathcal{P}: 2x - y + z = 3$ and perpendicularly intersects the line $l_1: x = 1+t, y = -1-t, z = 2+2t$.



$\vec{n} = \langle 2,-1,1\rangle$
$\vec{v}_1 = \langle 1,-1,2\rangle$

$\vec{n}\times\vec{v}_1 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 1 \\ 1 & -1 & 2 \end{vmatrix} = \langle -1,-3,-1\rangle$

We can take $\vec{v} = \langle 1,3,1\rangle$

P_0 is the intersection of l_1 with $\mathcal{P}$:

$2x-y+z=3 \Rightarrow 2(1+t)-(-1-t)+2+2t=3 \Rightarrow 5t=-2 \Rightarrow t=-2/5$

$\Rightarrow P_0 = (3/5, -3/5, 6/5)$.

Thus

$l: \begin{cases} x = \frac{3}{5}+t \\ y = \frac{-3}{5}+3t \\ z = \frac{6}{5}+t \\ t \in \mathbb{R} \end{cases}$ or $x - 3/5 = \frac{y+3/5}{3} = z - 6/5 \quad (=t)$

or

$\langle x,y,z\rangle = \langle \frac{3}{5}+t, \frac{-3}{5}+3t, \frac{6}{5}+t\rangle$ $t\in\mathbb{R}$

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Department of Mathematics

Group	CALCULUS WITH ANALYTIC GEOMETRY II MidTerm 2	List No.
Code : *Math 120* Acad. Year : *2012-2013* Semester : *Spring* Coordinator: *Muhiddin Uğuz* Date : *May.04.2013* Time : *9:30* Duration : *120 minutes*	Last Name : Name : Student No. : Department : Section : Signature : 7 QUESTIONS ON 4 PAGES TOTAL 100 POINTS	

1	2	3	4	5	6	7	SHOW YOUR WORK	

Question 1 (12 pts) Find the following limits, if they exist.

a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3y^3}{x^{12}+y^4}$

• Along x axis: $(y=0,\ x\to 0)$ $\displaystyle\lim_{x\to 0}\frac{x^3 0^3}{x^{12}+0^4} = \lim_{x\to 0}\frac{0}{x^{12}} = \lim_{x\to 0} 0 = 0$. Hence if given Limit exists, it is zero.

• Along the curve $y=x^3$: $\displaystyle\lim_{x\to 0}\frac{x^3(x^3)^3}{x^{12}+(x^3)^4} = \lim_{x\to 0}\frac{x^{12}}{2x^{12}} = \lim_{x\to 0}\frac{1}{2} = \frac{1}{2} \neq 0$

Hence limit does not exist.

b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2\sin y}{x^2+y^2}$

$$0 \le \left|\frac{x^2\sin y}{x^2+y^2}\right| = \frac{x^2}{x^2+y^2}|\sin y| \le |\sin y|$$

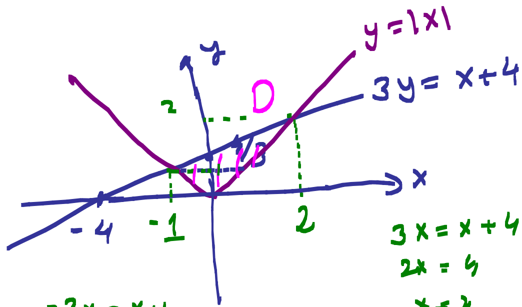
as $(x,y)\to(0,0)$: 0 and $|\sin y| \to 0$

. Hence by squeeze thm, $\displaystyle\lim_{(x,y)\to(0,0)} |f(x,y)| = 0$. Again by squeeze thm, $(-|f(x,y)| \le f(x,y) \le |f(x,y)|)$ $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

Question 2 (15 pts) Let D be the bounded region in the xy-plane bounded by $y=|x|$ and $3y = x+4$. Express (do **NOT** evaluate) the double integral $\displaystyle\int\int_D f(x,y)dA$ as an iterated integral with respect to

a) x first (in the order $dxdy$)

$$\int_0^1\int_{-y}^{y} f(x,y)\,dx\,dy + \int_1^2\int_{3y-4}^{y} f(x,y)\,dx\,dy$$



$3x = x+4$
$2x = 4$
$x = 2$

$-3x = x+4$
$-4x = 4$
$x = -1$

b) y first (in the order $dydx$)

$$\int_{-1}^{0}\int_{-x}^{\frac{x+4}{3}} f(x,y)\,dy\,dx + \int_0^2\int_x^{\frac{x+4}{3}} f(x,y)\,dy\,dx$$

OR $\displaystyle\int_{-1}^{2}\int_{|x|}^{\frac{x+4}{3}} f(x,y)\,dy\,dx$

Question 3 (6+6+3=15 pts) Let $f(x,y) = \begin{cases} \dfrac{xy+x^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

a) Find $f_x(0,0)$ and $f_y(0,0)$.

$$\lim_{t \to 0} \frac{f(t,0) - f(0,0)}{t} = \lim_{t \to 0} \frac{\frac{t^3}{t^2} - 0}{t} = \lim_{t \to 0} \frac{t^3}{t^3} = 1 = f_x(0,0)$$

$$\lim_{t \to 0} \frac{f(0,t) - f(0,0)}{t} = \lim_{t \to 0} \frac{\frac{0}{t^2} - 0}{t} = \lim_{t \to 0} \frac{0}{t^3} = \lim_{t \to 0} 0 = 0 = f_y(0,0)$$

b) Is f continuous at $(0,0)$?

$$\lim_{\substack{(x,y) \to (0,0) \\ \text{along } y=x}} f(x,y) = \lim_{x \to 0} \frac{x^2+x^3}{x^2+x^2} = \lim_{x \to 0} \frac{1+x}{2} = \frac{1}{2}$$

Hence if limit exists it must be $\frac{1}{2}$ but $f(0,0) = 0$

so

f is not continuous at $(0,0)$.

c) Find the function $f_x(x,y)$.

$$(x,y) \neq (0,0) \Rightarrow f_x(x,y) = \frac{(y+3x^2)(x^2+y^2) - 2x(xy+x^3)}{(x^2+y^2)^2} = \frac{yx^2+y^3+3x^4+3x^2y^2-2x^2y-2x^4}{(x^2+y^2)^2}$$

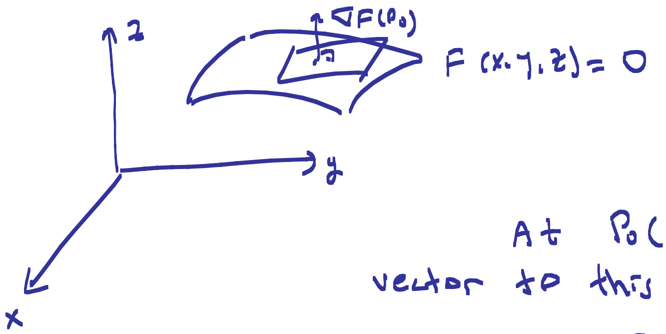
$$= \frac{x^4+y^3+3x^2y^2-x^2y}{(x^2+y^2)^2}$$

$(x,y) = (0,0) \Rightarrow f_x(0,0) = 1$ (from part (a))

Hence

$$f_x(x,y) = \begin{cases} \dfrac{x^4+y^3+3x^2y^2-x^2y}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0) \\ 1 & \text{if } (x,y) = (0,0) \end{cases}$$

Question 4 (14 pts) Find equation of the tangent plane drawn to the surface the surface $xy^2 + \sin(xyz) = zx + 1$ at the point $(1,1,0)$.



Let $F(x,y,z) = xy^2 + \sin(xyz) - zx - 1$

Then Given surface can be expressed as Level surface of F : $F(x,y,z) = 0$

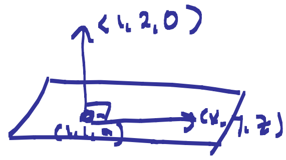
At $P_0(1,1,0)$, $\nabla F(P_0)$ can be used as normal vector to this level surface at P_0.

$F_x = y^2 + yz\cos(xyz) - z \Rightarrow F_x(1,1,0) = 1$

$F_y = 2xy + xz\cos(xyz) \Rightarrow F_y(1,1,0) = 2$

$F_z = xy\cos(xyz) - x \Rightarrow F_z(1,1,0) = 0$

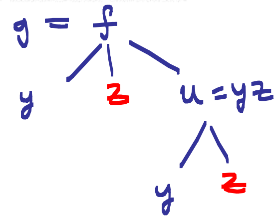
$\nabla F(P_0) = \langle 1, 2, 0 \rangle$



Tangent Plane: $\langle 1, 2, 0 \rangle \cdot \langle x-1, y-1, z \rangle = 0$

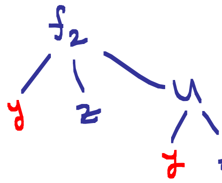
$x - 1 + 2y - 2 = 0 \Rightarrow x + 2y = 3$

Question 5 (12 pts) Let $g(y,z) = f(y,z,yz)$ where f has continuous second order partial derivatives.



a) Find $\dfrac{\partial g}{\partial z}$ in terms of partial derivatives (that is $f_1,\ f_2$, and f_3) of f .

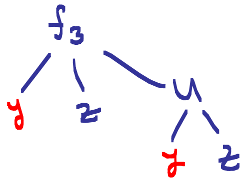
$$g_z(y,z) = f_2 + f_3 \cdot y$$



b) Find $\dfrac{\partial^2 g}{\partial y \partial z}$ in terms of partial derivatives of f.

$$\frac{\partial^2 g}{\partial y \partial z} = \frac{\partial}{\partial y}\left(\frac{\partial g}{\partial z}\right) = \frac{\partial}{\partial y}\left(f_2 + f_3 \cdot y\right)$$

$$= f_{21} + f_{23} \cdot z \quad + f_3 + y\left(f_{31} + f_{33} \cdot z\right)$$



Question 6 (3+8+6=17 pts)

Let $f(x,y) = xy$ and $D = \{(x,y) : x^2 + xy + y^2 \leq 12\}$ be the **region** bounded by an ellipse.

a) Does $f(x,y)$ have an absolute maximum-minimum on D? Explain.

Since $f(x,y) = xy$ being a polynomial in x & y is continuous on closed and bounded region D, it has absolute MAX./min on D.

b) Find the maximum-minimum values of $f(x,y) = xy$ on the boundary **curve** $x^2 + xy + y^2 = 12$ of D.

Max, min of $f(x,y) = xy$
subject to $g(x,y) = x^2 + xy + y^2 - 12 = 0$ } use Lagrange multipliers method:

set $\nabla f = \lambda \nabla g$:

$y \overset{(1)}{=} \lambda(2x + y)$
$x \overset{(2)}{=} \lambda(2y + x)$
$x^2 + y^2 + xy \overset{(3)}{=} 12$
} 3 unknowns: x, y, λ, 3 equations.

$(1) \Rightarrow y = 2\lambda x + \lambda y$
$(2) \Rightarrow x = 2\lambda y + \lambda x$
} $(y - x) = -2\lambda(y - x) + \lambda(y - x)$

$y \neq x$ $\Rightarrow$ $\lambda = -1$ $\Rightarrow$ $(1) \Rightarrow y = -2x - y \Rightarrow x = -y$

$(3) \Rightarrow x^2 + x^2 - x^2 = 12$
$x = \mp 2\sqrt{3}$

$f(2\sqrt{3}, -2\sqrt{3}) = -12 = f(-2\sqrt{3}, 2\sqrt{3})$

$y = x$ $\Downarrow (3)$ $x = y = 2$ or $x = y = -2$

$f(2,2) = f(-2,-2) = 4$

$\boxed{\text{Max} = 4, \ \text{min} = -12}$

c) Find the absolute maximum-minimum of $f(x,y)$ on the region D.

We checked the points on the boundary curve of D in part (b) and found that $f(2,2) = 4 = f(-2,-2)$

To check the points inside D: that is for $\{(x,y) : x^2 + xy + y^2 \lneq 12\}$

we need to check critical pts (since there is no singular point)

$f(x,y) = xy \Rightarrow \left.\begin{matrix} f_x = y \\ f_x = x \end{matrix}\right\} \Rightarrow \nabla f(x,y) = \vec{0} \iff (x,y) = (0,0)$ is the only critical point and $f(0,0) = 0$

$\therefore$ Absolute min of f on D is $f(2\sqrt{3}, -2\sqrt{3}) = f(-2\sqrt{3}, 2\sqrt{3}) = -12$
" MAX " " " " " $f(-2,-2) = f(2,2) = 4$

Question 7 (6+6+3=15 pts) Let $f(x,y) = \begin{cases} \dfrac{x^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

a) Find the directional derivative of f at $(0,0)$ in the direction of $\vec{v} = \vec{i} + \vec{j}$ by using the definition of directional derivative.

$\|\vec{v}\| = \sqrt{2}$, Unit vector in direction of $\vec{v}$ is $\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

$$\lim_{t\to 0} \frac{f(0+\frac{1\cdot t}{\sqrt{2}}, 0+\frac{1\cdot t}{\sqrt{2}}) - f(0,0)}{t} = \lim_{t\to 0} \frac{f(\frac{t}{\sqrt{2}}, \frac{t}{\sqrt{2}})}{t} = \lim_{t\to 0} \frac{\frac{\cancel{t^3}}{2\sqrt{2}}}{\cancel{t^2}\cdot\cancel{t}}$$

$$= \frac{1}{2\sqrt{2}}$$

b) Find $\vec{\nabla} f(0,0)$ and evaluate $\vec{\nabla} f(0,0) \cdot \dfrac{\vec{v}}{|\vec{v}|}$ (where $|\vec{v}|$ shows the length of $\vec{v}$).

- $$\lim_{t\to 0} \frac{f(t,0) - f(0,0)}{t} = \lim_{\substack{t\to 0 \\ (t\neq 0)}} \frac{\frac{t^3}{t^2}}{t} = \lim_{t\to 0} 1 = 1 = f_x(0,0)$$

- $$\lim_{t\to 0} \frac{f(0,t) - f(0,0)}{t} = \lim_{\substack{t\to 0 \\ (t\neq 0)}} \frac{\frac{0}{t^2} - 0}{t} = \lim_{t\to 0} 0 = 0 = f_y(0,0)$$

$$\nabla f(0,0) \cdot \frac{\vec{v}}{\|\vec{v}\|} = \langle 1, 0 \rangle \cdot \langle \tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}} \rangle = \frac{1}{\sqrt{2}}$$

c) Can f be differentiable at $(0,0)$? Explain.

f can not be differentiable at $(0,0)$. Because, if it were, we would have $D_{\vec{u}} f(0,0) = \nabla f(0,0) \cdot \vec{u}$, but as calculated in part (a) & (b) this is not the case.

(Note that f is continuous, f_x, f_y and $D_{\vec{u}} f$ all exist at $(0,0)$ but still f is not differentiable.)

M E T U
Department of Mathematics

Group	CALCULUS WITH ANALYTIC GEOMETRY II Final Exam	List No.

Code : *Math 120* Acad. Year : *2012-2013* Semester : *Spring* Coordinator: *Muhiddin Uğuz*	Last Name : Name : Student No. : Department : Section : Signature :
Date : *May.30.2013* Time : *9:30* Duration : *150 minutes*	7 QUESTIONS ON 6 PAGES TOTAL 100 POINTS

1	2	3	4	5	6	7	SHOW YOUR WORK	

Question 1 (3+3+6 pts) Let $S_n = \dfrac{n^2 - e^{-n}}{1+2n^2}$ be the sequence of partial sums of the series $\sum_{n=1}^{\infty} a_n$, $a_n \geq 0$.

a) Determine whether $\sum_{n=1}^{\infty} a_n$ is convergent or divergent.

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1 - \frac{1}{n^2 e^n}}{\frac{1}{n^2} + 2} = \frac{1-0}{2} = \frac{1}{2}$$

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n = \frac{1}{2}$$, so convergent

b) Find $\lim_{n\to\infty} a_n$.

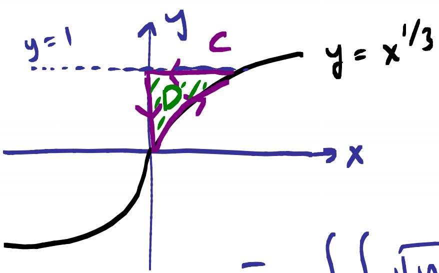
Since $\sum_{n=1}^{\infty} a_n$ is convergent, by n^{th} term test $\lim_{n\to\infty} a_n = 0$

c) Determine whether $\sum_{n=1}^{\infty} \dfrac{a_n}{1+a_n}$ is convergent or divergent.

$a_n \geq 0$. so $\dfrac{a_n}{1+a_n} \geq 0$. $\lim_{n\to\infty} \dfrac{\frac{a_n}{1+a_n}}{a_n} = \lim_{n\to\infty} \dfrac{1}{1+a_n} = \dfrac{1}{1+0} = 1$

$0 < 1 < \infty$, since $\sum_{n=1}^{\infty} a_n$ is convergent, by limit comparison test $\sum_{n=1}^{\infty} \dfrac{a_n}{1+a_n}$ is also convergent.

Question 2 (16 pts) Let D be the region bounded by the curves $y = x^{1/3}$, $x = 0$, and $y = 1$. Let C be the boundary of D, oriented counterclockwise. **Sketch** the region D and evaluate $\oint_C \vec{\mathbf{F}} \cdot d\vec{r}$ where $\vec{\mathbf{F}}(x,y) = \langle -y, x\sqrt{y^4+1}\rangle$ (that is $\vec{\mathbf{F}}(x,y) = -y\vec{i} + (x\sqrt{y^4+1})\vec{j}$).



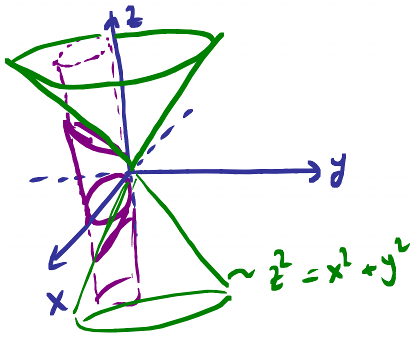
Note that D is simply connected & closed region. Moreover; $M(x,y) = -y$, $N(x,y) = x\sqrt{y^4+1}$ have continuous partial derivatives everywhere. $C = \partial D$ is positively oriented. So by Green's Theorem $\oint_C \langle M,N\rangle \cdot dr = \iint_D (N_x - M_y)\,dA$

$$= \iint_D \sqrt{y^4+1} + 1 \, dA = \int_0^1 \int_0^{y^3} \left(\sqrt{y^4+1} + 1\right) dx\, dy$$

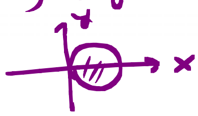
$$= \int_0^1 \left(y^3\sqrt{y^4+1} + y^3\right) dy = \left[\frac{1}{4}\,\frac{2}{3}\,(y^4+1)^{\frac{3}{2}} + \frac{y^4}{4}\right]_0^1$$

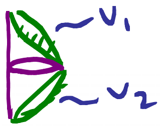
$$= \frac{1}{6}\,2^{3/2} + \frac{1}{4} - \frac{1}{6}$$

Question 3 (12 pts) Let S be the solid bounded by the cylinder $x^2 + y^2 = 2x$ and the cone $z^2 = x^2 + y^2$. **Sketch** S and find its volume.

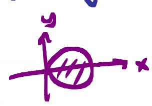


$x^2 - 2x + y^2 = 0 \Rightarrow (x-1)^2 + y^2 = 1$





$$V = V_1 + V_2 = 2V_1 = 2\iint_D \sqrt{x^2+y^2}\, dA$$



use polar coordinates:

$x^2 + y^2 = 2x \longleftrightarrow r = 2\cos\theta$

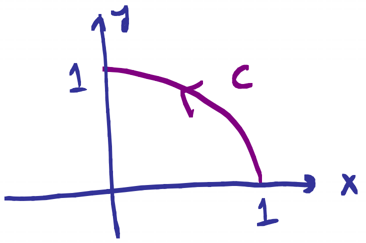
$$V = 4\int_0^{\pi/2}\int_0^{2\cos\theta} r \cdot r\, dr\, d\theta = \frac{4}{3}\int_0^{\pi/2} (2\cos\theta)^3\, d\theta = \frac{32}{3}\int_0^{\pi/2} \cos^3\theta\, d\theta$$

$$= \frac{32}{3}\int_0^{\pi/2} (1-\sin^2\theta)\cos\theta\, d\theta = \frac{32}{3}\left[\sin\theta - \frac{\sin^3\theta}{3}\right]_0^{\pi/2} = \frac{32}{3}\left[1 - \frac{1}{3}\right]$$

$$= \frac{32}{3}\cdot\frac{2}{3} = \frac{64}{9}$$

Question 4 (5+4+5+4 pts) Let C be the piece of the unit circle $x^2+y^2=1$, from $(1,0)$ to $(0,1)$ in counterclockwise direction.

a) For $\vec{\mathbf{G}}(x,y) = \langle xy, y\rangle$, evaluate $\int_C \vec{\mathbf{G}} \cdot d\vec{\mathbf{r}}$



$$C \leftrightarrow \vec{r}(t) = (\cos t, \sin t) \ ; \ t \in [0, \pi/2]$$

$$\vec{r}\,'(t) = \langle -\sin t, \cos t\rangle$$

$$\vec{G}(\vec{r}(t)) = \langle \cos t \sin t, \sin t\rangle$$

$$\vec{G}(\vec{r}(t)) \cdot \vec{r}\,'(t) = -\cos t \cdot \sin^2 t + \cos t \sin t$$

$$\int_C \vec{G} \cdot d\vec{r} = \int_0^{\pi/2} (-\cos t \sin^2 t + \cos t \sin t)\, dt = \left[-\frac{\sin^3 t}{3} - \frac{1}{4}\cos 2t \right]_0^{\pi/2}$$

$$= -\frac{1}{3} + \frac{1}{4} - \left(0 - \frac{1}{4}\right) = -\frac{1}{3} + \frac{1}{2} = \frac{1}{6}$$

b) Let $\vec{\mathbf{F}}(x,y) = \langle \cos x + 2xy,\ x^2 + \sin y\rangle$. **Without finding a potential function** show that $\vec{\mathbf{F}}$ is a conservative vector field on $\mathbb{R}^2$.

$$\frac{\partial}{\partial x}(x^2 + \sin y) = 2x, \qquad \frac{\partial}{\partial y}(\cos x + 2xy) = 2x .$$

Since $M(x,y) = \cos x + 2xy$ and $N(x,y) = x^2 + \sin y$ have continuous partials everywhere and $N_x = M_y$, $\vec{F}(x,y)$ is conservative.

c) Find a potential function $\phi(x,y)$ for the (conservative) field $\vec{\mathbf{F}}$ in part (b).

$$\phi_x = M = \cos x + 2xy \ \rightarrow\ \phi(x,y) = \sin x + x^2 y + f(y)$$

$$\Rightarrow N = x^2 + \sin y = \phi_y = x^2 + f'(y)$$

$$\Rightarrow f'(y) = \sin y \ \Rightarrow\ f(y) = -\cos y + c$$

$\therefore$ $\phi(x,y) = \sin x + x^2 y - \cos y + 120$ is a potential function for $\vec{F}$ because $\vec{\nabla}\phi = \vec{F}$.

d) Evaluate $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ where $\vec{\mathbf{F}}$ is the vector field in part (b).

Since $\vec{F} = \vec{\nabla}\phi$, $\int_C \vec{F} \cdot d\vec{r} = \phi(0,1) - \phi(1,0)$

$$= (-\cos 1 + \cancel{120}) - (\sin 1 - 1 + \cancel{120})$$

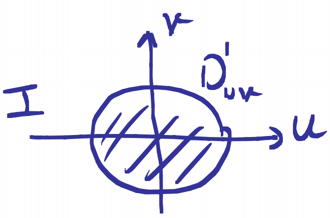
$$= 1 - \cos 1 - \sin 1$$

Question 5 (16 pts) Evaluate $\displaystyle\iint_D \frac{dA}{4x^2+9y^2+1}$ where D is the region bounded by the ellipse $(2x)^2+(3y)^2=1$. (Hint: Use a suitable change of variables).

Let $\left.\begin{matrix} u=2x \\ v=3y \end{matrix}\right\}$ Then $(2x)^2+(3y)^2=1$ becomes $u^2+v^2=1$

$$\frac{\partial(u,v)}{\partial(x,y)} = \det\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6$$

$$\iint_{D_{xy}} \frac{dA}{4x^2+9y^2+1} = \iint_{D'_{uv}} \frac{1}{u^2+v^2+1}\,\frac{1}{6}\,du\,dv = I$$

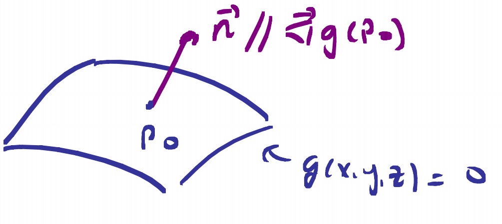


In polar coordinates $I = \frac{1}{6}\int_0^{2\pi}\int_0^1 \frac{1}{r^2+1}\, r\,dr\,d\theta$

$$= \frac{1}{6}(2\pi)\,\frac{1}{2}\ln(r^2+1)\Big|_0^1$$

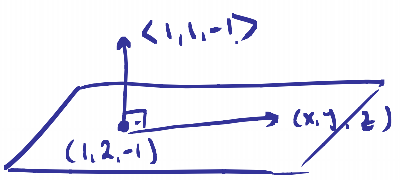
$$= \frac{\pi}{6}\cdot \ln 2.$$

Question 6 (10 pts) Find an equation for the tangent plane to the surface $g(x,y,z)=0$ where $g(x,y,z)=F(xyz,\ x^2+y^2+z^2)-5$ at the point $P_0(1,2,-1)$ if F is a differentiable function and $F(-2,6)=5,\ F_1(-2,6)=2,\ F_2(-2,6)=-1$.



- $g_x = F_1\, yz + F_2\, 2x$
  $\Rightarrow g_x(1,2,-1) = -2F_1(-2,6) + 2F_2(-2,6)$
  $= (-2)(2) + 2(-1) = -4-2 = -6$
- $g_y = F_1\, xz + F_2\, 2y \Rightarrow g_y(1,2,-1) = (-1)(2) + 4(-1)$
  $= -2-4 = -6$
- $g_z = F_1\, xy + F_2 \cdot 2z \Rightarrow g_z(1,2,-1) = 2(2) - 2(-1) = 4+2 =$
  $= 6$

Thus we can take $\vec{n} = \langle 1, 1, -1 \rangle$



$(x-1)+(y-2)-(z+1)=0$

$x+y-z=4$

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Question 7 (6+4+6 pts) Let $f(x, y) = e^{xy}(x^2 + y^2)$.

a) Find and classify all the critical points of f.

$f_x = e^{xy}(2x + y(x^2+y^2)) = 0 \Rightarrow 2x + x^2y + y^3 = 0$

$f_y = e^{xy}(2y + x(x^2+y^2)) = 0 \rightarrow 2y + y^2x + x^3 = 0$

multiplying 1^{st} equation by y & 2^{nd} by x we obtain

$$\left.\begin{array}{l} 2xy + x^2y^2 + y^4 = 0 \\ 2xy + x^2y^2 + x^4 = 0 \end{array}\right\} \Rightarrow x^4 = y^4 \Rightarrow x = \mp y$$

- $y = x \Rightarrow 2x + x^3 + x^3 = 0 = 2x(x^2+1) \Rightarrow x = 0 = y$
  $\rightarrow (0,0)$ is a critical point
- $y = -x \Rightarrow 2x - x^3 - x^3 = 0 = 2x(1-x^2) = 0 \Rightarrow x \in \{-1, 0, +1\}$
  $\Rightarrow (0,0), (-1,+1), (+1,-1)$ are critical points.

Thus The set of all critical points is $\{(0,0), (1,-1), (-1,1)\}$

$f_{xx} = e^{xy}(2 + 2xy + y(2x + x^2y + y^3))$
$= e^{xy}(2 + 4xy + x^2y^2 + y^4)$

$f_{yy} = e^{xy}(2 + 4xy + x^2y^2 + x^4)$

$f_{xy} = e^{xy}(x^2 + 3y^2 + x(2x + x^2y + y^3))$
$= e^{xy}(3x^2 + 3y^2 + x^3y + xy^3)$

$$\left.\begin{array}{l} \Delta(0,0) = 4 - 0 = 4 > 0 \\ f_{xx}(0,0) = 2 > 0 \end{array}\right\} (0,0) \text{ local min}$$

$\Delta(1,-1) = \Delta(-1,1) = 6 - \frac{16}{e^2} < 0$

$(1,-1)$ and $(-1,1)$ are saddle point.

b) Does $f(x, y)$ have an absolute maximum and minimum on $\mathbb{R}^2$? If so, find those values, if not explain why.

- $f(x,y) = e^{xy}(x^2+y^2) \geqslant 0 \quad \forall (x,y)$ and
  $f(0,0) = 0$
  so 0 is the minimum value.
- on $y = x$, $f(x,x) = 2x^2 e^{x^2} \rightarrow \infty$ as $x \rightarrow \infty$
  so f has <u>no</u> Maximum.

PLEASE TURN OVER

c) Find the absolute maximum and minimum of $f(x,y) = e^{xy}(x^2+y^2)$ on the region $x^2+y^2 \le 1$. Use the method of Lagrange multipliers on the boundary.

$(0,0)$ is the only critical point inside and $f(0,0) = 0$

To find Max/min of f on the boundary of the region, that is to find Max/min of $f(x,y) = e^{xy}(x^2+y^2)$ subject to $g(x,y) = x^2+y^2-1 = 0$] we Lagrange multipliers mtd.

Let $L(x,y,\lambda) = e^{xy}(x^2+y^2) + \lambda(x^2+y^2-1)$

$$\begin{cases} L_x = e^{xy}(2x + x^2y + y^3) + 2\lambda x \overset{(1)}{=} 0 \\ L_y = e^{xy}(2y + x^3 + xy^2) + 2\lambda y \overset{(2)}{=} 0 \\ L_\lambda = x^2+y^2-1 \overset{(3)}{=} 0 \end{cases}$$

$x = 0 \overset{(1)}{\Rightarrow} y = 0 \overset{(3)}{\Rightarrow} -1 = 0$ contradiction

$y = 0 \overset{(2)}{\Rightarrow} x = 0 \overset{(3)}{\Rightarrow} -1 = 0$ contradiction

so $x \neq 0$ & $y \neq 0$

since $x \neq 0$ & $y = 0$, we can write

$$\lambda = -e^{xy}\frac{(2x + x^2y + y^3)}{2x} = -e^{xy}\frac{(2y + x^3 + xy^2)}{2y}$$

$\Rightarrow 2xy + x^2y^2 + y^4 = 2xy + x^4 + x^2y^2 \Rightarrow y^4 = x^4$

$\Rightarrow y = \mp x \Rightarrow y^2 = x^2 \Rightarrow 2x^2 - 1 = 0 \Rightarrow x = \mp\frac{1}{\sqrt{2}}$

$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = f\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right) = \sqrt{e} \leftarrow$ MAX

$f\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right) = f\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{e}}$

inside the region: $f(0,0) = 0 \leftarrow$ min