M E T U
Department of Mathematics

	CALCULUS WITH ANALYTIC GEOMETRY MidTerm 1	
Code : *Math 120* Acad. Year : *2013-2014* Semester : *Spring* Coordinator: *Muhiddin Uğuz* Date : *April.5.2014* Time : *9:30* Duration : *120 minutes*	Last Name : Name : Student No. : Department : Section : Signature : 8 QUESTIONS ON 6 PAGES TOTAL 100 POINTS	

1	2	3	4	5	6	7	8	SHOW YOUR WORK	

Question 1 (8 pts) Suppose that $r \in \mathbb{R}$ and $a_n = nr^n$. For what values of r is the sequence $\{a_n\}$ convergent?

- if $r=0$, then $a_n = 0\ \forall n \Rightarrow \lim\limits_{n\to\infty} a_n = 0 \Rightarrow \{a_n\} = \{0\}$ is a convergent sequence
- If $|r| \geq 1$ then $\lim\limits_{n\to\infty} n.r^n$ does not exist since nr^n is unbounded
- If $\left[\begin{array}{l}|r|<1\\ r\neq 0\end{array}\right]$, consider $f(x) = x r^x$.

$$\lim_{x\to+\infty} |f(x)| = \lim_{x\to\infty} |x r^x| = \lim_{x\to\infty} \frac{x}{\left(\frac{1}{|r|}\right)^x} \quad \left(\tfrac{\infty}{\infty}\text{ type}\right)$$

$$\overset{\text{L'Hopital Rule}}{=} \lim_{x\to\infty} \frac{1}{\left(\frac{1}{|r|}\right)^x . \ln\left(\frac{1}{|r|}\right)} = 0 \Rightarrow \lim_{x\to\infty} f(x) = 0$$

$$\Rightarrow \lim_{n\to\infty} a_n = 0$$

so the sequence converges $\Longleftrightarrow |r| < 1$.

Question 2 (5+5=10 pts) Evaluate the sum of the infinite series

(a) $$\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{2^{3n}} = \frac{1}{8}\sum_{n=1}^{\infty}\left(\frac{-3}{8}\right)^{n-1} \underset{\left(\begin{array}{l}r=-3/8\\ |r|<1\end{array}\right)}{=} \frac{1}{8}\,\frac{1}{1-\left(-\frac{3}{8}\right)} = \frac{1}{11}$$

(b) $$\sum_{n=3}^{\infty} \frac{1}{n(n-2)} = \frac{1}{2}\sum_{n=3}^{\infty}\left(\frac{1}{n-2} - \frac{1}{n}\right)$$

$$\frac{1}{n(n-2)} = \frac{A}{n} + \frac{B}{n-2} = \frac{(A+B)n - 2A}{n(n-2)}$$

$$\Rightarrow \left.\begin{array}{l}A+B=0\\ -2A=1\end{array}\right\} \Rightarrow \begin{array}{l}A=-1/2\\ B=1/2\end{array}$$

$$= \frac{1}{2}\lim_{M\to\infty}\sum_{n=3}^{M}\left(\frac{1}{n-2}-\frac{1}{n}\right)$$

$$= \frac{1}{2}\lim_{M\to\infty}\left[\left(\frac{1}{1}-\cancel{\frac{1}{3}}\right)+\left(\frac{1}{2}-\cancel{\frac{1}{4}}\right)+\left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{5}}\right)+\left(\cancel{\frac{1}{4}}-\frac{1}{6}\right)+\cdots+\left(\cancel{\frac{1}{M-3}}-\frac{1}{M-1}\right)+\left(\cancel{\frac{1}{M-2}}-\frac{1}{M}\right)\right]$$

$$= \frac{1}{2}\lim_{M\to\infty}\left(\frac{3}{2}-\frac{1}{M-1}-\frac{1}{M}\right) = \frac{3}{4}$$

Question 3 (6+6+6=18 pts) Determine whether each series is convergent or divergent.

(a) $\sum_{n=1}^{\infty} \frac{\cos(3n)}{1+\left(\frac{120}{119}\right)^n} = \sum a_n$

consider $\sum |a_n|$. since $|a_n| = \frac{|\cos(3n)|}{1+\left(\frac{120}{119}\right)^n} < \frac{1}{\left(\frac{120}{119}\right)^n} = b_n$

since both $|a_n|$ & b_n are positive and

$\sum b_n$ is a geometric series (ie, of the form $\sum r^n$) with

$r = \frac{119}{120} < 1$ and thus convergent, by comparison test

$\sum |a_n|$ is also convergent.

so the original series $\sum a_n$ is (absolutely) convergent.

(b) $\sum_{n=1}^{\infty} \frac{2^n (n!)^2}{(2n)!} = \sum_{n=1}^{\infty} a_n$ where $a_n > 0$. We can apply ratio test:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\cancel{2}\cdot\cancel{2^n}\cdot(n+1)^{\cancel{2}}\cancel{(n!)^2}}{\cancel{(2n+2)}(2n+1)\cancel{(2n)!}} \cdot \frac{\cancel{(2n)!}}{\cancel{2^n}\cdot\cancel{(n!)^2}} = \lim_{n\to\infty} \frac{n+1}{2n+1}$$

$= \frac{1}{2} < 1 \Rightarrow \sum a_n$ is convergent.

(c) $\sum_{n=1}^{\infty} (\ln(2n-1) - \ln(n+5)) = \sum_{n=1}^{\infty} a_n$

$a_n = \ln(2n-1) - \ln(n+5) = \ln\left(\frac{2n-1}{n+5}\right)$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \ln\left(\frac{2n-1}{n+5}\right) = \ln 2 \neq 0$, hence by n^{th} Term $\sum_{n=1}^{\infty} a_n$ is divergent

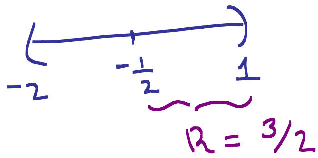
Question 4 (16 pts) Find the radius R, and the interval I of convergence for the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(2x+1)^n}{3^n\,\sqrt{n}}$

Apply Ratio Test;

$$\lim_{n\to\infty} \left| \frac{(2x+1)^{\cancel{n+1}}}{3^{\cancel{n+1}} \cdot \sqrt{n+1}} \right| \cdot \left| \frac{\cancel{3}^{n}\sqrt{n}}{\cancel{(2x+1)^n}} \right| = \frac{|2x+1|}{3} \lim_{n\to\infty} \sqrt{\frac{n}{n+1}} = \frac{|2x+1|}{3}$$

Thus

(i) if $\frac{|2x+1|}{3} < 1$ ie/ if $x \in (-2, 1)$, then given power series is convergent.



(ii) if $\frac{|2x+1|}{3} > 1$ ie/ if $x \in (-\infty, -2)$ or $(1, \infty)$ given power series is divergent

(iii) if $\frac{|2x+1|}{3} = 1$ ie/ if $x = 1$ or $x = -2$, then

$x = 1 \implies \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges by p-test $(p = \frac{1}{2})$

$x = -2 \implies \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ is an alternating series

$b_n = \frac{1}{\sqrt{n}}$ is decreasing, $b_n > 0$ and $\lim_{n\to\infty} b_n = 0$

so, converges by Alternating Series Test.

Thus $R = \frac{3}{2}$, $I = [-2, 1)$

Question 5 (6+6=12 pts)

(a) Compute the Maclaurin series of the indefinite integral $\int e^{-x^2}dx$ by using the known formula for the Maclaurin series of e^x.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \forall x \Rightarrow e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$$

Therefore Maclaurin series of $\int e^{-x^2}dx$ is $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{x^{2n+1}}{2n+1} + C \quad \forall x \in \mathbb{R}.$

(b) Find an approximation to the definite integral $\int_0^{1/10} e^{-x^2}dx$ such that the absolute value of the error is less than 10^{-6}.

$$\int_0^{1/10} e^{-x^2}dx \stackrel{\downarrow}{=} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{(2n+1)\ 10^{2n+1}} = \sum_{n=0}^{\infty} (-1)^n b_n \text{ where}$$

$b_n = \dfrac{1}{n!\,(2n+1)\ 10^{2n+1}} > 0$, b_n is decreasing, $\lim_{n\to\infty} b_n = 0$

so this is an alternating convergent series and absolute value of n^{th} error term is $< b_{n+1}$.

$$|E_n| < b_{n+1} = \frac{1}{(n+1)!\,(2n+3)\ 10^{2n+3}}.$$

If $n=1$, $b_2 = \dfrac{1}{2\cdot 5\cdot 10^5} = 10^{-6}$. $\Rightarrow |E_1| < 10^{-6}$

So we can set the approximation to be $(b_0 - b_1)$

$$\int_0^{1/10} e^{-x^2}dx \approx b_0 - b_1 = \frac{1/10}{0!\cdot 1} - \frac{1/10^3}{1!\cdot 3} = \frac{1}{10} - \frac{1}{3000} \approx 0.09966\ldots$$

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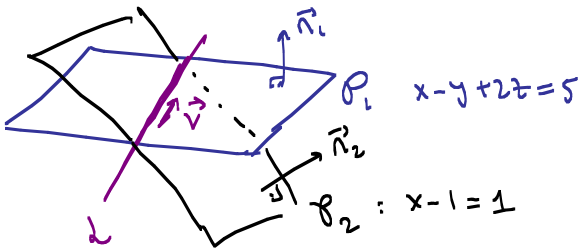
Question 6 (7+7=14 pts)

(a) Find parametric equations for the line which is the intersection of the planes $x - y + 2z = 5$ and $x - y = 1$.



$\vec{n}_1 = \langle 1, -1, 2\rangle$ and $\vec{n}_2 = \langle 1, -1, 0\rangle$ are normal vectors to the planes P_1 & P_2.

$\left.\begin{array}{l}\vec{v} \perp \vec{n}_1 \\ \vec{v} \perp \vec{n}_2\end{array}\right\} \Rightarrow \vec{v} \parallel \vec{n}_1 \times \vec{n}_2$

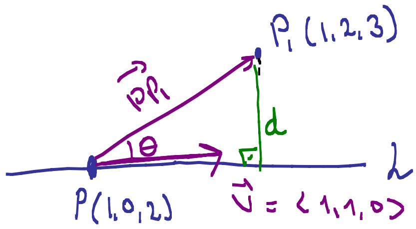
$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} i & j & k \\ 1 & -1 & 2 \\ 1 & -1 & 0 \end{vmatrix} = \langle 2, 2, 0\rangle$

we can take $\vec{v} = \langle 1, 1, 0\rangle$

since $P(1,0,2) \in P_1 \cap P_2 = L$, we can write $L: \vec{r}(t) = \langle 1,0,2\rangle + t\langle 1,1,0\rangle$ $t \in \mathbb{R}$

or $\left.\begin{array}{l} x = 1 + t \\ y = t \\ z = 2 \\ t \in \mathbb{R}\end{array}\right]$ parametric equation for the line L.

(b) Find the distance from the point $(1, 2, 3)$ to the line found in part (a).



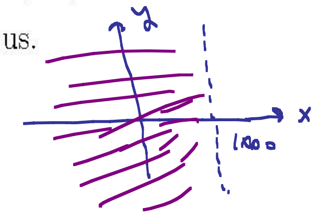
$\sin\theta = \frac{d}{\|\vec{PP_1}\|} \Rightarrow d = \|\vec{PP_1}\| \frac{\|\vec{v}\|}{\|\vec{v}\|} \sin\theta$

$d = \frac{\|\vec{PP_1} \times \vec{v}\|}{\|\vec{v}\|} = \frac{\sqrt{1+1+4}}{\sqrt{1+1+0}} = \frac{\sqrt{6}}{\sqrt{2}} = \sqrt{3}.$

$\vec{PP_1} \times \vec{v} = \begin{vmatrix} i & j & k \\ 0 & 2 & 1 \\ 1 & 1 & 0 \end{vmatrix} = \langle -1, 1, -2\rangle$

Question 7 (10 pts) Suppose that $f(x,y) = \begin{cases} 0, & \text{if } (x,y) = (0,0) \\ \dfrac{xy^2 \ln(1000 - x)}{x^2 + y^4}, & \text{if } (x,y) \neq (0,0). \end{cases}$

Find the domain of $f(x,y)$, and the set of all points at which $f(x,y)$ is continuous.



$\mathrm{Dom}(f(x,y)) = \{(x,y) : 1000 - x > 0\} = \{(x,y) ; x < 1000\}$

Clearly f is continuous if $(x,y) \in \mathrm{Dom}(f) \setminus \{(0,0)\}$

To check continuity of f at $(0,0)$; consider $\lim\limits_{(x,y)\to(0,0)} f(x,y)$

along $x = y^2$, $\lim\limits_{\substack{y\to 0 \\ (y\neq 0)}} f(y^2, y) = \lim\limits_{y\to 0} \frac{y^2 \cdot y^2 \ln(1000 - y^2)}{y^4 + y^4} = \frac{\ln(1000)}{2}$.

$\neq$ $\Rightarrow$ limit does not exist

along $y = 0$, $\lim\limits_{x\to 0} f(x, 0) = \lim\limits_{\substack{x\to 0 \\ (x\neq 0)}} \frac{0}{x^2 + 0} = \lim\limits_{x\to 0} 0 = 0$

$\Rightarrow f$ is not continuous at $(0,0) \in \mathrm{Dom}(f)$.

Therefore f is continuous on $\{(x,y) ; x < 1000, (x,y) \neq (0,0)\}$

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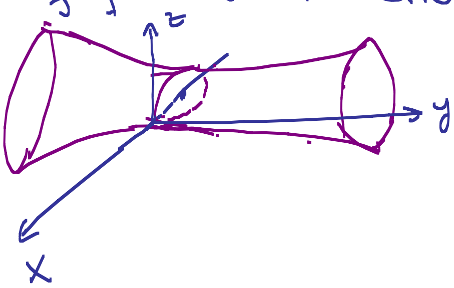
Question 8 (6+3+3=12 points)

(a) Consider the quadric surface $x^2 + 2x - 4y^2 + z^2 = 0$. Write the equations of the curves which are intersections of this surface with the planes parallel to xy, yz and xz planes and then identify these curves. (e.g. as circle, line, hyperbola, parabola, ellipse, etc.). Also state the name of the quadric surface and sketch it. Carefully label your axes in your drawing.

- intersection with planes $x=k$ (i.e. $1x+0y+0z=k$):
  $z^2 - 4y^2 = -k^2 - 2k$ "hyperbolas" on the plane $x=k$
- Intersection with planes $y=k$: $x^2+2x+z^2 = 4k^2$
  $(x+1)^2 + z^2 = 4k^2 + 1$ "ellipse" on the plane $y=k$
- Intersection with planes $z=k$: $x^2 + 2x - 4y^2 = -k^2$
  $(x+1)^2 - 4y^2 = 1-k^2$ "hyperbolas" on the plane $z=k$

so the figure is a one sheeted hyperboloid



$(x+1)^2 - 4y^2 + z^2 = 1$

(b) Show that the curves $\mathbf{r}(t) =< 0, t, -2t >$ where $t \in \mathbb{R}$ and $\mathbf{u}(s) =< -2s^2, -s^2, 2s >$ where $s \in \mathbb{R}$ both lie on the surface in part (a).

$0^2 + 2\cdot 0 - 4t^2 + 4t^2 = 0$ ✓

$4s^4 - 4s^2 + 4s^4 + 4s^2 = 0$ ✓

(c) Show that the two curves in part (b) intersect at the origin, and find the cosine of the angle θ between them at this intersection point, where $0 \leq \theta \leq \pi/2$. (Recall that the angle between two smooth curves at a point is equal to the angle between their tangent lines at that point).

$\vec{r}(0) = \vec{u}(0) = \langle 0,0,0\rangle$ so they intersect at the origin.

$\vec{r}\,'(t) = \langle 0, 1, -2\rangle \Rightarrow \vec{r}\,'(0) = \langle 0, 1, -2\rangle$

$\vec{u}\,'(s) = \langle -4s, -2s, 2\rangle \Rightarrow \vec{u}\,'(0) = \langle 0, 0, 2\rangle$

$$\cos\theta = \frac{\langle 0,1,-2\rangle \cdot \langle 0,0,2\rangle}{\|\langle 0,1,-2\rangle\| \; \|\langle 0,0,2\rangle\|} = \frac{-4}{\sqrt{5}\cdot 2} = -\frac{2}{\sqrt{5}}$$

but then $\theta > \frac{\pi}{2}$ $\Rightarrow$ complement $\pi - \theta$ has

$\cos(\pi - \theta) = \frac{2}{\sqrt{5}}$

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Department of Mathematics

	CALCULUS WITH ANALYTIC GEOMETRY MidTerm 2	
Code : *Math 120* Acad. Year : *2013-2014* Semester : *Spring* Coordinator: *Muhiddin Uğuz* Date : *May.3.2014* Time : *9:30* Duration : *120 minutes*	Last Name : Name : Student No. : Department : Section : Signature : 6 QUESTIONS ON 4 PAGES TOTAL 100 POINTS	

1	2	3	4	5	6	SHOW YOUR WORK	

Question 1 (10+6 pts) Let $u(x,y,z) = x\,f\left(\frac{y}{x},\frac{z}{x}\right)$ where f has continuous partial derivatives of all orders.

(a) Compute u_x and u_y in terms of partial derivatives of f.

$$u_x = f\left(\tfrac{y}{x},\tfrac{z}{x}\right) + x\left[f_1\left(\tfrac{y}{x},\tfrac{z}{x}\right)\left(\tfrac{-y}{x^2}\right) + f_2\left(\tfrac{y}{x},\tfrac{z}{x}\right)\left(\tfrac{-z}{x^2}\right)\right] = f - \frac{y}{x}f_1 - \frac{z}{x}f_2$$

$$u_y = x\left[f_1\frac{1}{x} + f_2\cdot 0\right] = f_1$$

(b) Compute u_{xz} in terms of partial derivatives of f.

$$u_{xz} = \frac{\partial}{\partial z}\left[f - \frac{y}{x}f_1 - \frac{z}{x}f_2\right] = f_1\cdot 0 + \cancel{f_2\cdot\frac{1}{x}} - \frac{y}{x}\left(f_{11}\,0 + f_{12}\frac{1}{x}\right) - \cancel{\frac{1}{x}f_2} - \frac{z}{x}\left(f_{21}\cdot 0 + f_{22}\frac{1}{x}\right)$$

$$= \frac{-y}{x^2}f_{12} - \frac{z}{x^2}f_{22} = \frac{-1}{x^2}\left(y f_{12} + z f_{22}\right)$$

Question 2 (5+5+8 pts)

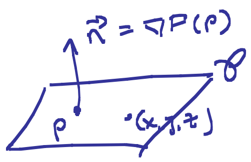
(a) Find a vector $\vec{n}$ normal to the surface $x^2+4y^2+10z^2 = 30$ at the point $P(2,2,1)$.

Surface is given as level surface of $F(x,y,z) = x^2+4y^2+10z^2$ and hence $\nabla F(P)$ is normal to surface at P.

$\nabla F(x,y,z) = \langle F_x, F_y, F_z\rangle = \langle 2x, 8y, 20z\rangle \Rightarrow \nabla F(P) = \langle 4, 16, 20\rangle$ is a vector normal to given surface at P.

(or any vector of the form $\langle k, 4k, 5k\rangle$, $k \in \mathbb{R}\setminus\{0\}$)

(b) Write an equation of the tangent plane to the surface in part (a) at the point $P(2,2,1)$.



$(x,y,z) \in \mathcal{P} \iff \vec{n}\cdot\langle x-2, y-2, z-1\rangle = 0$

$\iff x-2+4y-8+5z-5 = 0$

$x+4y+5z = 15$

(c) Find the directional derivative of $f(x,y,z) = 2014 - x^2 - 4y^2 - 10z^2$ at the point $P(2,2,1)$ in the direction of $\vec{n}$ which is found in part (a).

$D_{\vec n} f(2,2,1) = \nabla f(2,2,1)\cdot\frac{\vec n}{\|\vec n\|} = \langle -4,-16,-20\rangle\cdot\langle 4,16,20\rangle\frac{1}{\sqrt{672}} = -\sqrt{672}$

$\nabla f = \langle -2x, -8y, -20z\rangle$

$\nabla f(P) = \langle -4, -16, -20\rangle$

(or, if you take $\vec n$ to be $\langle k, 4k, 5k\rangle$ for some $k<0$, then, $D_{\vec n} f(2,2,1) = +\sqrt{672}$)

Question 3 (15 pts) Use Lagrange Multipliers Method to find the point(s) on the paraboloid $2z = x^2 + 2y^2 - 5$ which is closest to the origin.

Find the point(s) on $2z = x^2 + 2y^2 - 5$ that minimize $f(x,y,z) = x^2+y^2+z^2$
(they also minimize $d(x,y,z) = \sqrt{x^2+y^2+z^2}$)

$\Rightarrow$ Find $\left[\begin{array}{l}\text{min of } f(x,y,z) = x^2+y^2+z^2 \\ \text{subject to } g(x,y,z) = x^2+2y^2-2z-5=0\end{array}\right]$

Use Lagrange Multipliers Method; set $\nabla f = \lambda \nabla g$:

$$\left[\begin{array}{l} f_x = 2x \overset{(1)}{=} 2\lambda x = \lambda g_x \\ f_y = 2y \overset{(2)}{=} 4\lambda y = \lambda g_y \\ f_z = 2z \overset{(3)}{=} -2\lambda = \lambda g_z \\ 2z \overset{(4)}{=} x^2+2y^2-5 \end{array}\right]$$

4 unknowns
4 equations

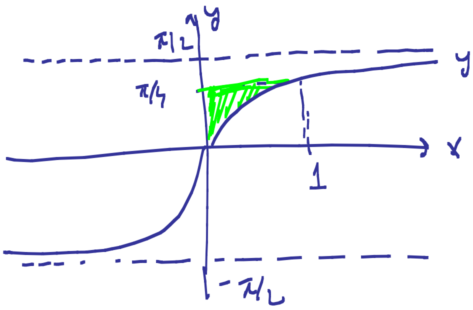
$x = 0 \Rightarrow y$:
- $y = 0 \Rightarrow z = -\frac{5}{2} \Rightarrow \bullet P_1\left(0, 0, -\frac{5}{2}\right)$
- $y \neq 0 \overset{(2)}{\Longrightarrow} \lambda = \frac{1}{2} \overset{(3)}{\Longrightarrow} z = -\frac{1}{2} \overset{(4)}{\Longrightarrow} y = \mp\sqrt{2} \Rightarrow \left[\begin{array}{l}\bullet P_2\left(0, -\sqrt{2}, -\frac{1}{2}\right) \\ \bullet P_3\left(0, \sqrt{2}, -\frac{1}{2}\right)\end{array}\right.$

$x \neq 0 \overset{(1)}{\Longrightarrow} \lambda = 1 \left\{\begin{array}{l} \overset{(3)}{\Longrightarrow} z = -1 \\ \overset{(2)}{\Longrightarrow} y = 0 \end{array}\right\} \overset{(4)}{\Longrightarrow} -2 = x^2 - 5 \Rightarrow x = \mp\sqrt{3} \Rightarrow \left[\begin{array}{l}\bullet P_4\left(\sqrt{3}, 0, -1\right) \\ \bullet P_5\left(-\sqrt{3}, 0, -1\right)\end{array}\right.$

$$f(P_2) = f(P_3) = 2 + \frac{1}{4} = \frac{9}{4} < f(P_4) = f(P_5) = 4 < f(P_1) = \frac{25}{4}$$

so P_2 & P_3 are the points on the paraboloid $2z = x^2+2y-5$ that are closest to the origin.

Question 4 (15 pts) Evaluate the iterated integral $\displaystyle\int_0^1 \int_{\arctan(x)}^{\pi/4} e^{\sec(y)} \sec(y)\, dy dx$.



$$= \iint_R e^{\sec y} \sec y \, dA$$

$$= \int_0^{\pi/4} \int_0^{\tan y} e^{\sec y} \sec y \, dx \, dy$$

$$= \int_0^{\pi/4} e^{\sec y} \cdot \sec y \cdot \tan y \, dy = e^{\sec y} \Big|_0^{\pi/4} = e^{\sec \pi/4} - e^{\sec 0}$$

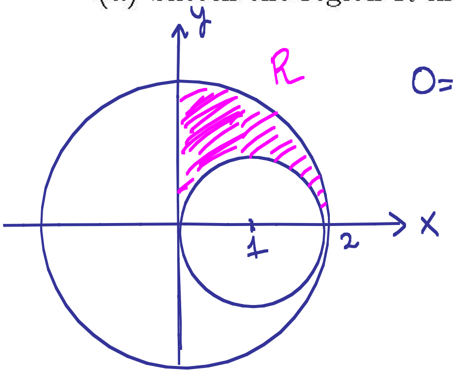
$$= e^{\sqrt{2}} - e$$

$\sec\frac{\pi}{4} = \frac{1}{\cos \pi/4} = \frac{2}{\sqrt{2}} = \sqrt{2}$

$\sec 0 = \frac{1}{\cos 0} = \frac{1}{1} = 1$

Question 5 (4+8+6 pts) Let R be the region which lies in the first quadrant $(\{(x,y);\ x \geq 0,\ y \geq 0\})$ between two circles: $x^2+y^2=4$ and $x^2-2x+y^2=0$.

(a) Sketch the region R in the xy-plane and express it in terms of polar coordinates.



$$0 = x^2-2x+y^2 = x^2-2x+1-1+y^2 \Rightarrow (x-1)^2+y^2=1$$

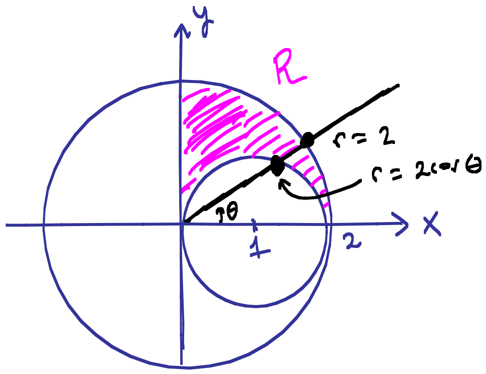
$$0 = x^2+y^2-2x = r^2-2r\cos\theta \Rightarrow r=0 \text{ or } r=2\cos\theta$$

so

$$R = \{(r,\theta)\ ;\ \theta\in[0,\tfrac{\pi}{2}],\ 2\cos\theta \leq r \leq 2\}$$

(b) Write and then evaluate the double integral of $f(x,y)=xy$ on R using **polar** coordinates.

$$\iint_R f(x,y)\,dA = \int_0^{\pi/2}\int_{2\cos\theta}^{2} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta = \int_0^{\pi/2}\int_{2\cos\theta}^{2} r^3\cos\theta\sin\theta\,dr\,d\theta$$



$$= \int_0^{\pi/2} \cos\theta\sin\theta \left.\frac{r^4}{4}\right|_{r=2\cos\theta}^{r=2} d\theta$$

$$= \frac{1}{4}\int_0^{\pi/2} \cos\theta\sin\theta\,(16-16\cos^4\theta)\,d\theta$$

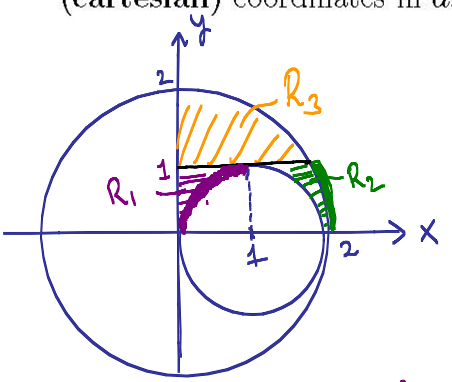
$$= 4\left[\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta - \int_0^{\pi/2}\cos^5\theta\sin\theta\,d\theta\right]$$

$u=\sin\theta$, $du=\cos\theta\,d\theta$ $\qquad$ $u=\cos\theta$, $du=-\sin\theta\,d\theta$

$$= 4\left[\int_0^1 u\,du + \int_1^0 u^5\,du\right] = 4\left[\frac{u^2}{2}-\frac{u^6}{6}\right]_0^1$$

$$= 4\left[\frac{1}{2}-\frac{1}{6}\right] = \frac{4}{3}$$

(c) Write the integral in part (b) as an iterated integral or sum of integrals in **rectangular (cartesian)** coordinates in $dxdy$ order. (**Do not evaluate !**)



$$\iint_R xy\,dA = \iint_{R_1} xy\,dA + \iint_{R_2} xy\,dA + \iint_{R_3} xy\,dA$$

$$R = R_1 \cup R_2 \cup R_3$$

$$= \int_0^1\int_0^{1-\sqrt{1-y^2}} xy\,dx\,dy + \int_0^1\int_{1+\sqrt{1-y^2}}^{\sqrt{4-y^2}} xy\,dx\,dy + \int_1^2\int_0^{\sqrt{4-y^2}} xy\,dx\,dy$$

$$(x-1)^2+y^2=1 \Rightarrow (x-1)^2 = 1-y^2$$
$$\Rightarrow x = 1 \mp \sqrt{1-y^2}$$

$$x^2+y^2=4 \Rightarrow x = \mp\sqrt{4-y^2}$$

$$\left[\text{OR}/\ \int_0^2\int_0^{\sqrt{4-y^2}} xy\,dx\,dy - \int_0^1\int_{1-\sqrt{1-y^2}}^{1+\sqrt{1-y^2}} xy\,dx\,dy\right]$$

Question 6 (8+10 pts) Let D be the solid region bounded by sphere $x^2+y^2+z^2=2$ and the paraboloid $z=x^2+y^2$.

(a) **Write** the volume of D as an iterated triple integral in **rectangular (cartesian)** coordinates. (**Do not evaluate!**)

$x^2+y^2+z^2=2$ & $z=x^2+y^2$

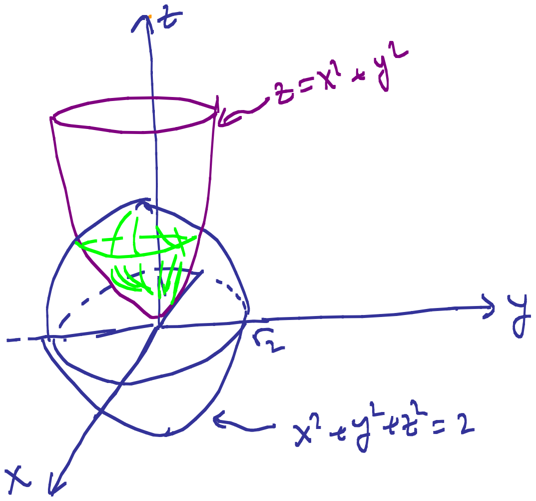
$z^2+z-2=0$

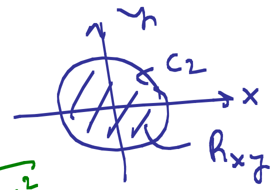
$(z+2)(z-1)=0$ & $z>0$

$\Rightarrow z=1$

$C_1 = \{(x,y,z) ; x^2+y^2=1, z=1\}$

$C_2 = \{(x,y,z) ; x^2+y^2=1, z=0\}$





$$\therefore \iiint_D 1 \, dV = \int_{-1}^{+1} \int_{-\sqrt{1-x^2}}^{+\sqrt{1-x^2}} \int_{x^2+y^2}^{\sqrt{2-x^2-y^2}} 1 \, dz\, dy\, dx$$

(If you take the solid region inside the sphere, below the paraboloid you also get full credit)

(b) **Compute** the volume of the solid region D using double integral in **polar** coordinates.

$x^2+y^2+z^2=2 \Rightarrow z=\mp\sqrt{2-r^2}$

$z=x^2+y^2 \Rightarrow z=r^2$

$$\text{Volume} = \iint_{R_{xy}} \left(\sqrt{2-r^2} - r^2\right) r \, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^1 r\sqrt{2-r^2}\, dr\, d\theta - \int_0^{2\pi} \int_0^1 r^3 \, dr\, d\theta$$

$u = 2-r^2$

$du = -2r\, dr$

$$= 2\pi \left[\frac{-1}{2} \int_2^1 u^{1/2} du \right] - 2\pi \left. \frac{r^4}{4} \right|_0^1$$

$$= 2\pi \left[\frac{1}{\cancel{2}} \frac{\cancel{2}}{3} u^{\frac{3}{2}} \Big|_1^2 - \frac{1}{4} \right] = 2\pi \left[\frac{1}{3}(\sqrt{8}-1) - \frac{1}{4} \right]$$

$$= 2\pi \left[\frac{2\sqrt{2}}{3} - \frac{4}{12} - \frac{3}{12} \right] = 2\pi \left[\frac{2\sqrt{2}}{3} - \frac{7}{12} \right] = \frac{\pi}{6}(8\sqrt{2}-7)$$

M E T U
Department of Mathematics

	CALCULUS WITH ANALYTIC GEOMETRY Final	

Code : *Math 120* Acad. Year : *2013-2014* Semester : *Spring* Coordinator: *Muhiddin Uğuz* Date : *May.29.2014* Time : *9:30* Duration : *150 minutes*	Last Name : Name : Student No. : Department : Section : Signature :
	7 QUESTIONS ON 6 PAGES TOTAL 100 POINTS

1	2	3	4	5	6	7	SHOW YOUR WORK	

Question 1 (9+6=15 pts)

(a) Find the MacLaurin series for $\tan^{-1}(x)$ by integrating the MacLaurin series of $\dfrac{1}{1+x^2}$. Find the largest open interval I on which the function $\tan^{-1}(x)$ is equal to its MacLaurin series. (Notation: $\tan^{-1}(x) = \arctan(x)$.)

Recall that $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n \ \forall\ |x|<1$. Hence $\dfrac{1}{1+x^2} = \dfrac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$

$$\tan^{-1}x = \arctan x = \int_0^x \frac{1}{1+t^2}\,dt = \int_0^x \sum_{n=0}^{\infty} (-1)^n t^{2n}\,dt$$

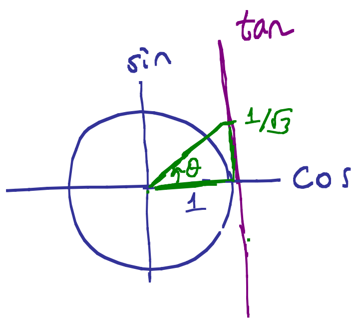
$$= \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{2n+1}\Big|_0^x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \qquad \forall\ |-x^2|<1 \quad \text{ie}\ \forall\ |x|<1$$

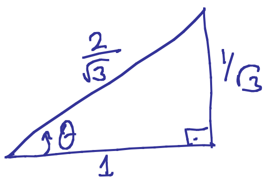
$$\therefore\ \tan^{-1}x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \qquad \forall\ x \in (-1,+1) = I$$

(b) Using the MacLaurin series you found in part (a), explicitly find the sum of the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\left(\frac{1}{\sqrt{3}}\right)^{2n+1} = \theta$$

$$\theta = \tan^{-1}\left(\frac{+1}{\sqrt{3}}\right) \Longrightarrow \left[\begin{array}{l} \tan\theta = \frac{1}{\sqrt{3}} \\ \text{and} \\ \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \end{array}\right.$$





$\Longrightarrow\ \theta = \pi/6$

Question 2 (15 pts) Find all local maximum, local minimum values and saddle points of the function $f(x,y) = e^y(y^2 - x^2)$.

$f_x(x,y) = e^y(-2x)$

$f_y(x,y) = e^y[(y^2-x^2)+2y]$

f_x & f_y always exist.

$f_x = 0 = f_y \xRightarrow{e^y \neq 0} x = 0$ & $y^2+2y = y(y+2) = 0 \Rightarrow y=0$ or $y=-2$

Set of all critical pts is $\{P_1(0,0), P_2(0,-2)\}$

$f_{xx} = -2e^y$

$f_{yy} = e^y[(y^2-x^2+2y)+2y+2]$

$f_{xy} = -2xe^y$

$D(x,y) = f_{xx}(x,y) f_{yy}(x,y) - f_{xy}^2(x,y)$

at $P_1(0,0)$

$D(0,0) = -2 \cdot 2 - 0^2 = -4 < 0 \Rightarrow f(0,0) = 0$ is saddle point.

at $P_1(0,-2)$

$f_{xx}(0,-2) = -2e^{-2}$

$f_{yy}(0,-2) = e^{-2}[4-4-4+2]$
$= -2e^{-2}$

$f_{xy}(0,-2) = 0$

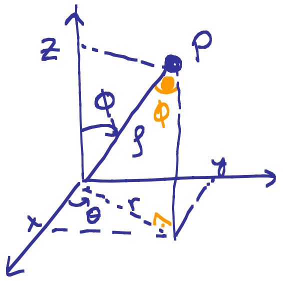
$D(0,-2) = (-2e^{-2})^2 - 0^2$
$= 4\frac{1}{e^4} > 0$

$f_{xx}(0,-2) = -2e^{-2} < 0$

$\Rightarrow f(0,-2) = \frac{4}{e^2}$ is local MAX

Question 3 (5+10=15 pts) Let (ρ, ϕ, θ) denote spherical coordinates. Consider the solid bounded below by the sphere $x^2 + y^2 + z^2 = 2z$ and above by the cone $z = \sqrt{x^2 + y^2}$.

(a) Show that the equation of the sphere above in spherical coordinates is $\rho = 2\cos\phi$.



$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ \sin\phi &= \frac{r}{\rho} \\ \cos\phi &= \frac{z}{\rho} \end{aligned}\right\} \quad \left.\begin{aligned} x &= \rho\sin\phi\cos\theta \\ y &= \rho\sin\phi\sin\theta \\ z &= \rho\cos\phi \end{aligned}\right\} \quad x^2+y^2+z^2=\rho^2$$

Thus $x^2+y^2+z^2 = 2z$ corresponds to

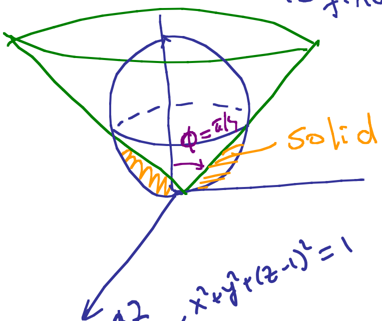
$\rho^2 = 2\rho\cos\phi$ and since $\rho \neq 0$ at the points on the sphere

$\rho = 2\cos\phi$

(b) Find the volume of this solid using a triple integral in spherical coordinates.

$x^2+y^2+z^2-2z+1 = 1 \implies x^2+y^2+(z-1)^2 = 1$ is sphere centered at $(0,0,1)$ with radius 1

· To find intersection: $x^2+y^2+(x^2+y^2) = 2\sqrt{x^2+y^2} \Rightarrow (x^2+y^2) = \sqrt{x^2+y^2} \Rightarrow \sqrt{x^2+y^2} = 1$



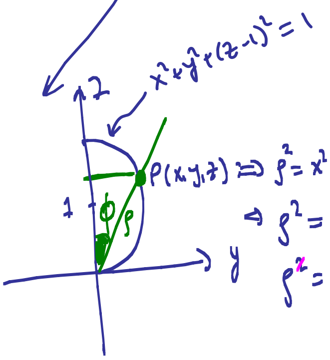
$$\text{Volume} = \iiint_S 1\,dV = \iiint_S 1\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^{2\cos\phi} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta = 2\pi\int_{\pi/4}^{\pi/2} \sin\phi\,\frac{\rho^3}{3}\Big|_0^{2\cos\phi} d\phi$$

$$= \frac{2\pi}{3}8\int_{\pi/4}^{\pi/2}\cos^3\phi\,\sin\phi\,d\phi = \frac{16\pi}{3}\int_0^{\sqrt{2}/2} u^3\,du = \frac{16\pi}{3}\,\frac{u^4}{4}\Big|_0^{\sqrt{2}/2}$$

$u = \cos\phi$

$du = -\sin\phi\,d\phi$



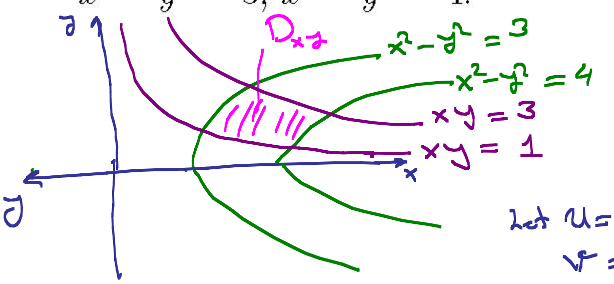
$P(x,y,z) \Rightarrow \rho^2 = x^2+y^2+z^2$

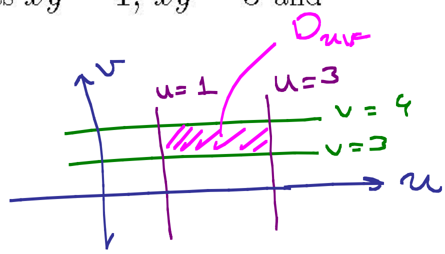
$\Rightarrow \rho^2 = 2z$

$\rho^{\cancel{2}} = 2\cancel{\rho}\cos\phi$

$$= \frac{\cancel{16}\pi}{3\cdot\cancel{4}}\cdot\frac{\cancel{4}}{\cancel{16}} = \pi/3$$

Question 4 (10 pts) Evaluate the double integral $\iint_D (x^4 - y^4)e^{xy}\,dA$ where D is the region in the first quadrant enclosed by the hyperbolas $xy = 1$, $xy = 3$ and $x^2 - y^2 = 3$, $x^2 - y^2 = 4$.





Let $u = xy : 1 \to 3$

$v = x^2-y^2 : 3 \to 4$

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}} = \frac{1}{\det\begin{bmatrix} y & x \\ 2x & -2y \end{bmatrix}} = \frac{1}{-2y^2-2x^2} = \frac{-1}{2(x^2+y^2)}$$

$$\iint_{D_{xy}} f(x,y)\,dy\,dx = \iint_{D_{uv}} f(x(u,v), y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dv\,du = \frac{1}{2}\int_1^3\int_3^4 v\,e^u\,dv\,du$$

$$= \frac{1}{2}\int_1^3 e^u\,\frac{v^2}{2}\Big|_3^4 du = \frac{1}{4}(4^2-3^2)\int_1^3 e^u\,du = \frac{7}{4}(e^3-e)$$

Question 5 (5+7+3=15 pts) Consider the vector field $\mathbf{F}$ defined on $\mathbb{R}^2$ given by

$$\mathbf{F}(x,y) = (\underbrace{2x\cos y + y\cos x}_{M(x,y)})i + (\underbrace{\sin x - x^2\sin y}_{N(x,y)})j.$$

(a) Determine whether or not the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path on the whole xy-plane.

Since M & N have continuous partial derivatives on all of $\mathbb{R}^2$ which is simply connected the necessary condition for F to be conservative (line integral of vector field F is independent of path), that is $M_y = N_x$ is also sufficient.

$M_y = -2x\sin y + \cos x$
$N_x = \cos x - 2x\sin y$ $\Big\}$ $M_y = N_x$ and hence $\int_C F\cdot dr$ is independent of path on $\mathbb{R}^2$.

OR, Let's try to find a potential function $f(x,y)$ for F (ie $\nabla f = F$)

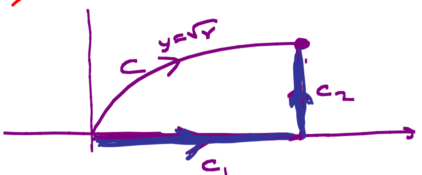
$f_x = M = 2x\cos y + y\cos x \Rightarrow f(x,y) = x^2\cos y + y\sin x + c(y) \Rightarrow f_y = -x^2\sin y + \sin x + c'(y)$

$f_y = N = \sin x - x^2\sin y \Rightarrow c'(y) = 0 \Rightarrow c(y) = c$. Thus $f(x,y) = x^2\cos y + y\sin x + c$ is a potential function for F, Hence F is conservative on all of $\mathbb{R}^2$ which is simply connected. so given line integral is independent of path.

(b) Compute the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ where C is the curve $y = \sqrt{x}$, $0 \le x \le 1$.

By part (a);
$$\int_C F\cdot dr = \int_C \nabla f\cdot dr = f(\text{terminal pt}) - f(\text{initial pt})$$
$$= f(1,1) - f(0,0) = \cos 1 + \sin 1$$

OR



Since $\int_C F\cdot dr$ is independent of path, we have
$$\int_C F\cdot dr = \int_{C_1} F\cdot dr + \int_{C_2} F\cdot dr$$

$C_1 \longleftrightarrow r_1(t) = (t,0)$; $t\in[0,1]$
$r_1'(t) = \langle 1,0\rangle$

$C_2 \longleftrightarrow r_2(t) = (1,t)$: $t\in[0,1]$
$r_2'(t) = \langle 0,1\rangle$

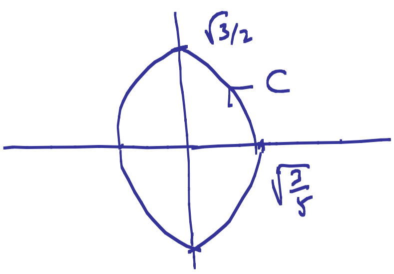
$\langle 2x\cos y + y\cos x,\ \sin x - x^2\sin y\rangle$

$$\int_{C_1} F\cdot dr = \int_0^1 F(r_1(t))\cdot r_1'(t)\,dt$$
$$= \int_0^1 \langle 2t, \sin t\rangle\cdot\langle 1,0\rangle\,dt = \int_0^1 2t\,dt = t^2\Big|_0^1 = 1 = A$$

$$\int_{C_2} F\cdot dr = \int_0^1 F(r_2(t))\cdot r_2'(t)\,dt$$
$$= \int_0^1 \langle 2\cos t + t\cos 1,\ \sin 1 - \sin t\rangle\cdot\langle 0,1\rangle\,dt = \sin 1 + \cos 1 - 1 = B$$

$A + B = \sin 1 + \cos 1$.

(c) Compute the line integral $\oint_C \mathbf{F}\cdot d\mathbf{r}$ where C is the ellipse $5x^2 + 2y^2 = 3$ oriented in the counter-clockwise direction.



Since $\oint_C F\cdot dr$ is independent of path on all of $\mathbb{R}^2$ and C is a closed curve (initial pt = terminal pt)

we have
$$\oint_C F\cdot dr = 0$$

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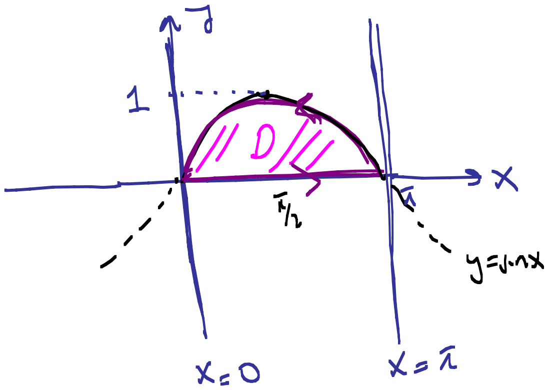
Question 6 (15 points) By using Green's theorem, evaluate the line integral

$$I = \oint_C \underbrace{e^x(1-\cos y)}_{M(x,y)}dx - \underbrace{e^x(120-\sin y)}_{N(x,y)}dy$$

where C is the boundary of the region $D = \{(x,y) | 0 < x < \pi, 0 < y < \sin x\}$ oriented counterclockwise.



$N_x = -e^x(120 - \sin y)$

$M_y = e^x \sin y$

Notice that M & N have continuous partials on all of $\mathbb{R}^2$ (and hence on D)

Moreover

$$N_x - M_y = -120e^x + \cancel{e^x}\sin y - \cancel{e^x}\sin y$$
$$= -120e^x$$

and C is positively oriented closed curve bounding simply connected region R; hence we can use green's thm:

$$\oint_{C=\partial R} M\,dx + N\,dy \overset{\text{Green's Thm}}{=} \iint_R (N_x - M_y)\,dA = \int_0^\pi \int_0^{\sin x} -120e^x\,dy\,dx$$

$$= -120\int_0^\pi e^x \sin x\,dx = -\frac{120}{2} e^x(\sin x - \cos x)\Big|_0^\pi$$

$$= -60\left[e^\pi(0-(-1)) - e^0(0-1)\right] = -60\left[e^\pi + 1\right]$$

$$\boxed{\int \underbrace{e^x}_{u}\,\underbrace{\sin x\,dx}_{dv}} = -e^x\cos x + \int \underbrace{e^x}_{u}\,\underbrace{\cos x\,dx}_{dv} = -e^x\cos x + e^x\sin x - \boxed{\int e^x \sin x\,dx}$$

$du = e^x\,dx$

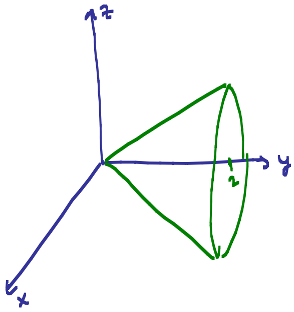
$v = -\cos x$

$du = e^x\,dx$

$v = \sin x$

Thus $\int e^x \sin x\,dx = \frac{e^x}{2}(\sin x - \cos x) + C$

Question 7 (15 points) Let S be the part of the cone $y^2 = x^2 + z^2$ bounded by the planes $y = 0$ and $y = 2$. Find a parametrization of S and use this parametrization to compute its surface area.



$S = r(\overset{r}{\overset{\shortparallel}{u}}, \overset{\theta}{\overset{\shortparallel}{v}}) = r(r,\theta) = (r\cos\theta, \overset{r}{\overset{\shortparallel}{\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}}}, r\sin\theta)$

$(r,\theta) \in [0,2] \times [0,2\pi] = R$

Surface Area $= SA = \iint_R \|r_r \times r_\theta\| \, dA$

$r_r = \langle \cos\theta, 1, \sin\theta \rangle$

$r_\theta = \langle -r\sin\theta, 0, r\cos\theta \rangle$

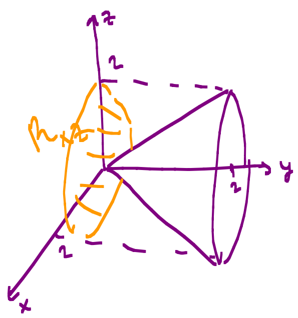
$r_r \times r_\theta = \langle r\cos\theta, \overset{-r}{\overset{\shortparallel}{-r\sin^2\theta - r\cos^2\theta}}, r\sin\theta \rangle$

$\|r_r \times r_\theta\|^2 = r^2\cos^2\theta + r^2 + r^2\sin^2\theta = 2r^2$

$SA = \int_0^{2\pi}\int_0^2 \sqrt{2}\cdot r \, dr\, d\theta = \cancel{2}\pi\cdot\sqrt{2}\,\frac{r^2}{\cancel{2}}\Big|_0^2 = 4\pi\sqrt{2}$

OR

This cone is graph of the function $y = f(x,z) = \sqrt{x^2+z^2}$
for $(x,z) \in R_{xz} = \{(x,z) ; x^2+z^2 \le 2\}$



Thus we may take
$r(x,z) = (x, \sqrt{x^2+z^2}, z)$ as a parametrization
$(x,z) \in R_{xz}$

Then $r_x = (1, \frac{x}{\sqrt{x^2+z^2}}, 0)$

$r_z = (0, \frac{z}{\sqrt{x^2+z^2}}, 1)$

$r_x \times r_z = \left(\frac{x}{\sqrt{x^2+z^2}}, -1, \frac{z}{\sqrt{x^2+z^2}}\right)$

$\|r_x \times r_z\| = \sqrt{\frac{x^2+z^2}{x^2+z^2}+1} = \sqrt{2}$

Hence $SA = \iint_{R_{xz}} \|r_x \times r_z\| \, dA = \iint_{R_{xz}} \sqrt{2}\, dA = \sqrt{2}\,.\,\text{Area}(R_{xz})$
$= \sqrt{2}\,.\,4.\pi$