M E T U
Department of Mathematics

	Calculus of Functions of Several Variables Final Exam	
Code : MATH 120 Acad. Year: 2014-2015 Semester : Spring Coord. : M.Uğuz Date : 28.05.2015 Time : 9.30 Duration : 135 minutes	Last Name : Name : Student No : Department: Section No : Signature :	
	6 Questions on 6 Pages Total 100 Points	

1	2	3	4	5	6	SHOW YOUR WORK !	

Q.1 (10+5= **15 pts**) Recall the geometric series (GS) : $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ for $|r| < 1$.

(a) Use GS to find the Maclaurin series of the function $f(x) = 2\ln(1 + \frac{1}{2}x^4)$. Find the interval of convergence of this Maclaurin series.

$$\ln(1+x) = \int_0^x \frac{1}{1+r}\,dr = \int_0^x \frac{1}{1+(-r)}\,dr = \int_0^x \sum_{n=0}^{\infty} (-1)^n r^n\,dr = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} \quad \forall |x|<1$$

Thus

$$f(x) = 2\ln(1+\tfrac{1}{2}x^4) = 2\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)} \frac{1}{2^{n+1}} x^{4n+4} \quad \forall \left|\tfrac{1}{2}x^4\right| < 1 \iff |x|^4 < 2 \iff |x| < 2^{1/4}$$

check end pts $x = \mp 2^{1/4}$: $2\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)\cancel{2^{n+1}}} \cancel{2^{n+1}} = 2\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ is convergent alternating harmonic series.

(also since GS is divergent for $|r|>1$, so is it's integral: Hence $2\sum \frac{(-1)^n}{(n+1)} \frac{1}{2^{n+1}} (x^4)^{n+1}$ is divergent if $|x|>2^{1/4}$)

Thus

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n\,(n+1)} (x^4)^{n+1} \quad \forall |x| \le 2^{1/4}$$

$$= x^4 - \frac{1}{4}x^8 + \frac{1}{12}x^{12} - \cdots$$

(b) Use PART (a) to evaluate the limit: $\lim_{x\to 0} \frac{f(x) - x^4}{x^8}$

$$\lim_{x\to 0} \frac{\left(x^4 - \frac{1}{4}x^8 + \frac{1}{12}x^{12} - \cdots\right) - x^4}{x^8} = \lim_{x\to 0} \frac{-\frac{1}{4}x^8 + \frac{1}{12}x^{12} - \cdots}{x^8} = \frac{-1}{4}$$

Q.2 (5 + 10 = **15 pts**)

(a) Let $\mathcal{C}$ be the space curve defined by the vector function $\vec{r}(t) = \langle t^3, \mathrm{e}^t, \ln(1+5t)\rangle$ for $t > -1/5$. Find parametric equations of the line ℓ, which is **tangent** to curve $\mathcal{C}$ at the point $(0, 1, 0)$.

$\vec{r}(t) = (0,1,0) = (t^3, e^t, \ln(1+5t)) \Rightarrow t = 0$

so direction vector of the tangent line is $\vec{r}\,'(0)$ where

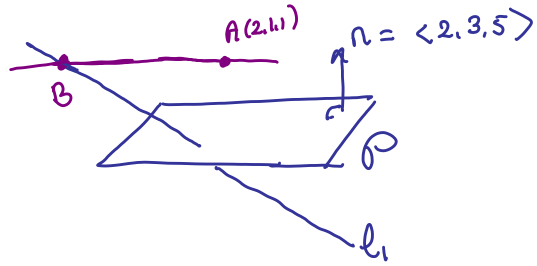
$\vec{r}\,'(t) = \langle 3t^2, e^t, \frac{5}{1+5t}\rangle$. so $\vec{r}\,'(0) = \langle 0, 1, 5\rangle$

Hence parametric equation of tangent line is

$$\begin{cases} x = 0 + 0t \\ y = 1 + 1t \\ z = 0 + 5t \\ t \in \mathbb{R} \end{cases} \qquad \begin{cases} x = 0 \\ y = 1 + t \\ z = 5t \\ t \in \mathbb{R} \end{cases}$$

(b) Let $\mathcal{P}: \; 2x + 3y + 5z = 120$ and $\ell_1: \dfrac{x-1}{1} = \dfrac{y-4}{-2} = \dfrac{z-4}{3}$ be a **plane** and a **line** in $\mathbb{R}^3$, respectively. Find parametric equations of the line ℓ_2 which has the properties:

- ℓ_2 passes through the point $A(2, 1, 1)$
- ℓ_2 is parallel to plane $\mathcal{P}$
- ℓ_2 intersects with line ℓ_1.



$\ell_2: \begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \\ t \in \mathbb{R} \end{cases}$

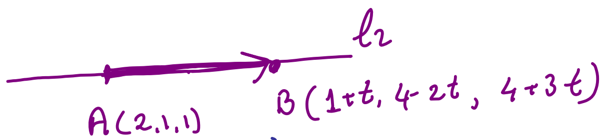
since ℓ_2 passes through A, we can take $(x_0, y_0, z_0) = A$

Hence $\ell_2: \begin{cases} x = 2 + at \\ y = 1 + bt \\ z = 1 + ct \end{cases}$

Let $\ell_1 \cap \ell_2 = B$. Then since $B \in \ell_1$, we have

$B = (1+t, 4-2t, 4+3t)$ for some $t \in \mathbb{R}$

$\ell_1: \begin{cases} x = 1 + t \\ y = 4 - 2t \\ z = 4 + 3t \\ t \in \mathbb{R} \end{cases}$



we can take $\vec{AB} = \langle -1+t, 3-2t, 3+3t\rangle$ as a direction vector for ℓ_2, which must be perpendicular to normal vector $\vec{n} = \langle 2, 3, 5\rangle$ of plane $\mathcal{P}$

Thus we must have

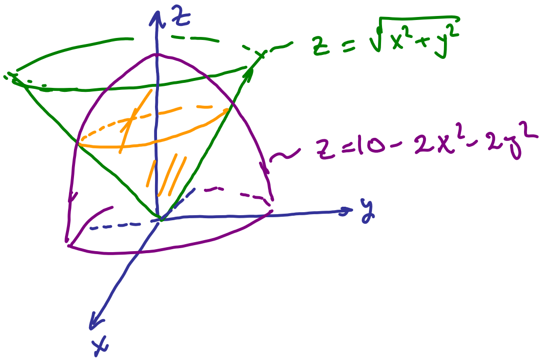
$\vec{AB} \cdot \vec{n} = -2 + 2t + 9 - 6t + 15 + 15t = 22 + 11t = 0 \Rightarrow t = -2$

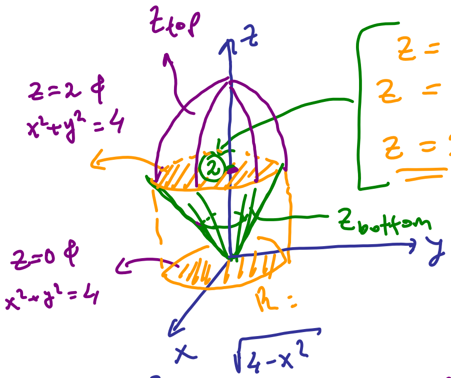
$\Rightarrow \vec{AB} = \langle -3, 7, -3\rangle$

Thus $\ell_2: \begin{cases} x = 2 - 3t \\ y = 1 + 7t \\ z = 1 - 3t \\ t \in \mathbb{R} \end{cases}$

Q.3 (5 + 5 + 10 = **20 pts**) Let Ω be the solid region in $\mathbb{R}^3$ bounded by the quadric surfaces $z = \sqrt{x^2+y^2}$ and $z = 10 - 2x^2 - 2y^2$.

(a) Express the VOLUME of Ω as an iterated DOUBLE integral in Cartesian coordinates (x, y). Do NOT evaluate the integral.





$$z = 10 - 2(x^2+y^2)$$
$$z = 10 - 2z^2$$
$$\underline{z = 2} \quad \text{or} \quad z = \cancel{-\tfrac{5}{2}} < 0$$

$$\text{Volume} = \iint_R (z_{top} - z_{bottom})\, dA = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left(10 - 2x^2 - 2y^2 - \sqrt{x^2+y^2}\right) dy\, dx$$

(b) Express the VOLUME of Ω as an iterated TRIPLE integral in Cartesian coordinates (x, y, z). Do NOT evaluate the integral.

$$V = \iiint_\Omega \underbrace{f(x,y,z)}_{\equiv 1}\, dV = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{10-2x^2-2y^2} 1 \; dz\, dy\, dx$$

(c) Express the VOLUME of Ω as an iterated TRIPLE integral in Cylindrical coordinates (r, θ, z). Then **evaluate** the integral.

$$V = \iiint_\Omega 1\, r\, dz\, dr\, d\theta = \int_0^{2\pi} \int_0^{2} \int_r^{10-2r^2} r\, dz\, dr\, d\theta$$

$$= 2\pi \int_0^2 r\, z \Big|_{z=r}^{z=10-2r^2} dr = 2\pi \int_0^2 r(10 - 2r^2 - r)\, dr = 2\pi \int_0^2 (10r - 2r^3 - r^2)\, dr$$

$$= 2\pi \left(\cancel{10}^{\,5} \frac{r^2}{\cancel{2}} - \frac{\cancel{2} r^4}{\cancel{4}_{\,2}} - \frac{r^3}{3} \Bigg|_0^2 \right) = 2\pi \left[20 - 8 - \frac{8}{3} \right] = \frac{56}{3}\pi$$

Q.4 ($3 \times 5 =$ **15 pts**) Let C be the plane curve defined by $y = \frac{1}{2}x^2 - \frac{1}{4}\ln x$ for $x \in [1, \mathrm{e}]$. Let $f(x, y)$ be a real-valued (scalar) continuous function for $x > 0$, $y \in \mathbb{R}$.

(a) Express the line integral $\int_C f(x,y)\mathrm{d}s$ as a **definite integral**.

$$C \longleftrightarrow \vec{r}(t) = \langle x(t), y(t) \rangle = \langle t, \tfrac{1}{2}t^2 - \tfrac{1}{4}\ln t \rangle \; : \; t \in [1, e]$$

is a parametrization for C.

$$\vec{r}\,'(t) = \langle 1, t - \tfrac{1}{4t} \rangle \Rightarrow \|\vec{r}\,'(t)\| = \sqrt{1 + (t - \tfrac{1}{4t})^2} = \sqrt{(t + \tfrac{1}{4t})^2}$$
$$= |t + \tfrac{1}{4t}| = t + \tfrac{1}{4t} \quad \forall t \in [1, e]$$

Hence

$$\int_C f(x,y)\, ds = \int_1^e f(\vec{r}(t))\, \|\vec{r}\,'(t)\|\, dt$$
$$= \int_1^e f\left(t, \tfrac{t^2}{2} - \tfrac{1}{4}\ln t\right)\left(t + \tfrac{1}{4t}\right) dt$$

(b) Express the **arclength** of curve C as a **line integral**, and then evaluate it.

$$\text{Length}(C) = L = \int_C \underbrace{f(x,y)}_{\overset{\shortparallel}{1}} \, ds = \int_1^e \left(t + \tfrac{1}{4t}\right) dt = \left[\frac{t^2}{2} + \frac{1}{4}\ln|t|\right]\Big|_1^e$$
$$= \frac{e^2}{2} + \frac{\overset{\shortparallel\,1}{\ln e}}{4} - \frac{1}{2} - \frac{1}{4}\cancel{\ln 1}^{\,0} = \frac{e^2}{2} - \frac{1}{4}$$

(c) If $f(x, y) = \left(x + \dfrac{1}{4x}\right)^{-1} x^{5/4} \mathrm{e}^y$, evaluate $\int_C f(x,y)\mathrm{d}s$.

$$\int_C f\, ds = \int_1^e \cancel{\left(t + \tfrac{1}{4t}\right)^{-1}}\, t^{5/4}\, e^{\frac{1}{2}t^2 - \frac{1}{4}\ln t}\, \cancel{\left(t + \tfrac{1}{4t}\right)}\, dt$$
$$= \int_1^e t^{\frac{5}{4}}\, e^{\frac{1}{2}t^2 - \frac{1}{4}\ln t}\, dt = \int_1^e t^{\frac{5}{4}} \cdot t^{-\frac{1}{4}} \cdot e^{\frac{1}{2}t^2}\, dt$$
$$= \int_1^e t\, e^{\frac{1}{2}t^2}\, dt = e^{\frac{1}{2}t^2}\Big|_1^e = e^{\frac{1}{2}e^2} - e^{\frac{1}{2}}$$

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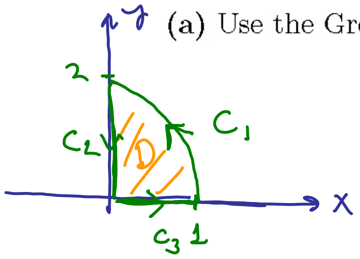
Q.5 (7 + 8 = **15 pts**) Let $\mathcal{C}$ be the counterclockwise oriented **boundary curve** of the plane region $\mathcal{D} = \{(x,y) \in \mathbb{R}^2 : x^2 + \frac{1}{4}y^2 \leq 1,\ x \geq 0,\ y \geq 0\}$.



(a) Use the Green's theorem to evaluate $\oint_{\mathcal{C}} (\cos x - y^2)\mathrm{d}x + 2y\ln(1+x)\mathrm{d}y$.

$C = C_1 + C_2 + C_3$

$F(x,y) = \langle M(x,y), N(x,y)\rangle = \langle \cos x - y^2, 2y\ln(1+x)\rangle$ has components that have continuous partial derivatives on D. $\partial D = C = C_1 + C_2 + C_3$ is piecewise smooth, positively oriented closed curve. Hence we can use green's thm:

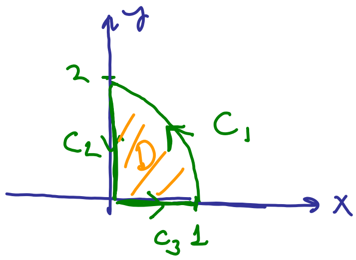
$$\oint_{C=\partial R} M\,dx + N\,dy = \iint_R (N_x - M_y)\,dA = \int_0^1 \int_0^{2\sqrt{1-x^2}} \left[\frac{2y}{1+x} - (-2y)\right] dy\,dx$$

$$= \int_0^1 \int_0^{2\sqrt{1-x^2}} 2y\left(\frac{2+x}{1+x}\right) dy\,dx = \int_0^1 y^2\left(\frac{2+x}{1+x}\right)\Big|_{y=0}^{y=2\sqrt{1-x^2}} dx$$

$$= \int_0^1 4(1-x^2)\frac{(2+x)}{(1+x)}\,dx = 4\int_0^1 (2-x-x^2)\,dx = \frac{14}{3}$$

(b) Let $\mathcal{C}_1$ be the **part** of the ellipse $x^2 + \frac{1}{4}y^2 = 1$ in the **first** quadrant oriented from $(1,0)$ to $(0,2)$. Use the result of PART (a) to evaluate

$$W = \int_{\mathcal{C}_1} (\cos x - y^2)\mathrm{d}x + 2y\ln(1+x)\mathrm{d}y.$$



$$\frac{14}{3} = \int_C M\,dx + N\,dy = \overbrace{\int_{C_1} M\,dx + N\,dy}^{W} + \int_{C_2} M\,dx + N\,dy + \int_{C_3} M\,dx + N\,dy$$

so $W = \frac{14}{3} - \int_{C_2} M\,dx + N\,dy - \int_{C_3} M\,dx + N\,dy$

$C_2 \leftrightarrow r_2(t) = (0, 2-t) : t \in [0,2]$

$C_3 \leftrightarrow r_3(t) = (t, 0) ; t \in [0,1]$

$$W = \frac{14}{3} - \int_0^2 \langle \cos 0 - (2-t)^2, 2(2-t)\overbrace{\ln 1}^{0}\rangle \cdot \langle 0, -1\rangle\,dt - \int_0^1 \langle \cos t - 0^2, 2\cdot 0 \ln(1+t)\rangle \cdot \langle 1, 0\rangle\,dt$$

$$= \frac{14}{3} - \int_0^2 0\,dt - \int_0^1 \cos t\,dt = \frac{14}{3} - \sin t\,\Big|_0^1 = \frac{14}{3} - \sin(1)$$

Q.6 (4 × 5 = **20 pts**) Let $\vec{F}(x,y,z)$ be a vector field in the space $\mathbb{R}^3$ defined by

$$\vec{F}(x,y,z) = \langle 2x + y^2 + z\cos x \,,\, 2xy + e^z \,,\, 1 + ye^z + \sin x \rangle.$$

Let C be the curve parametrized by $\vec{r}(t) = \langle x,y,z \rangle = \langle t^2, t^4, t^{3/2} \rangle$ for $t \in [0,1]$.

(a) Express $\int_C \vec{F} \bullet d\vec{r}$ as a **definite integral** but do NOT evaluate it.

$$\vec{r}\,'(t) = \langle 2t, 4t^3, \tfrac{3}{2}t^{1/2} \rangle$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt =$$

$$= \int_0^1 \left[(2t^2 + t^8 + t^{3/2}\cos t^2)\,2t + (2t^6 + e^{t^{3/2}})\,4t^3 + (1 + t^4 e^{t^{3/2}} + \sin t^2)\tfrac{3}{2}t^{1/2}\right]dt$$

(b) Find a potential $\Phi(x,y,z)$ to show that $\vec{F}(x,y,z)$ is **conservative** in $\mathbb{R}^3$.

$$\nabla\Phi(x,y,z) = \langle \Phi_x(x,y,z), \Phi_y(x,y,z), \Phi_z(x,y,z) \rangle = F(x,y,z) = \langle M(x,y,z), N, P \rangle$$

$\Rightarrow \Phi_x = M = 2x + y^2 + z\cos x \;\Rightarrow\; \Phi = x^2 + xy^2 + z\sin x + c(y,z)$ for some function of y & z

$\Rightarrow \Phi_y = 2xy + c_y(y,z)$

$\Phi_y = N \;\Rightarrow\; 2xy + c_y(y,z) = 2xy + e^z \;\Rightarrow\; c_y(y,z) = e^z \;\Rightarrow\; c(y,z) = ye^z + h(z)$

$\Rightarrow \Phi(x,y,z) = x^2 + xy^2 + z\sin x + ye^z + h(z)$

$\Rightarrow \Phi_z = \sin x + ye^z + h'(z) = P = 1 + ye^z + \sin x$

$\Rightarrow h'(z) = 1 \;\Rightarrow\; h(z) = z + c$

Thus

$\Phi(x,y,z) = x^2 + xy^2 + z\sin x + ye^z + z + c$ is a potential function for $\vec{F}$ for any real number c.

(c) Use the **potential function** $\Phi(x,y,z)$ to evaluate $\int_C \vec{F} \bullet d\vec{r}$.

$$\int_C \vec{F}\cdot d\vec{r} = \Phi(\text{terminal point of } C) - \Phi(\text{initial point of } C)$$

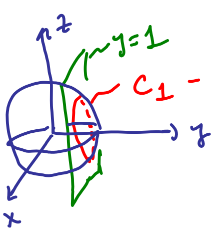
$$= \Phi(r(1)) - \Phi(r(0)) = \Phi(1,1,1) - \Phi(0,0,0)$$

$$= 1 + 1 + \sin 1 + e + 1$$

$$= 3 + e + \sin 1$$

(d) If C_1 is the curve of **intersection** of the sphere $x^2 + y^2 + z^2 = 9$ and the plane $y = 1$, evaluate $\int_{C_1} \vec{F} \bullet d\vec{r}$.

$$\int_{C_1} \vec{F}\cdot d\vec{r} = \oint_{C_1} \vec{F}\cdot d\vec{r} = \underline{0}$$



C_1 - close curve & F is conservative.

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Department of Mathematics

Calculus of Functions of Several Variables **Second Midterm Exam**		
Code : MATH 120 Acad. Year: 2014-2015 Semester : Spring Coord. : M.Uğuz Date : 02.05.2015 Time : 9.30 Duration : 135 minutes	Last Name : Name : Student No : Department: Section No : Signature :	
	6 Questions on **6** Pages Total **100** Points	

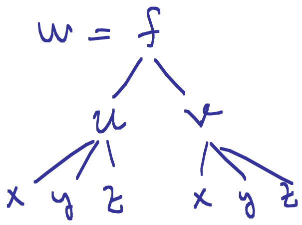
1	2	3	4	5	6	
						SHOW YOUR WORK !

Q.1 (8 + 7 = **15 pts**) Let $w = w(x, y, z)$ be the function defined by

$$w = f(u, v) = f\left(y\sin(xz),\, e^{xyz}\right)$$

where f is an arbitrary function with continuous second-order partial derivatives.

(a) Find $w_x = \dfrac{\partial w}{\partial x}$ and $w_y = \dfrac{\partial w}{\partial y}$ in terms of the partial derivatives f_u and f_v of f.

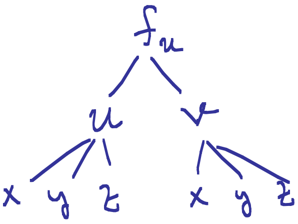


$$w_x = f_u(u,v)\, u_x(x,y,z) + f_v(u,v)\, v_x(x,y)$$
$$= f_u(u,v)\, yz\cos(xz) + f_v(u,v)\, yz\, e^{xyz}$$

similarly,

$$w_y = f_u \cdot u_y + f_v\, v_y$$
$$= f_u \cdot \sin(xz) + f_v\, xz\, e^{xyz}$$

(b) Find $w_{yz} = \dfrac{\partial}{\partial z}\left(\dfrac{\partial w}{\partial y}\right) = \dfrac{\partial^2 w}{\partial z \partial y}$ in terms of the partial derivatives f_u, f_v, f_{uu}, f_{uv} and f_{vv} of f.



$$w_y = f_u \cdot \sin(xz) + f_v\, xz\, e^{xyz}$$

$$(w_y)_z = (f_u)_z \sin(xz) + f_u \cdot x\cos(xz) + x\left[(f_v)_z\, z\, e^{xyz} + f_v \cdot e^{xyz} + f_v \cdot z\, xy\, e^{xyz}\right]$$

$$= (f_{uu} \cdot u_z + f_{uv}\, v_z)\sin(xz) + f_u \cdot x\cos(xz) + x\, e^{xyz}\left[(f_{vu} \cdot u_z + f_{vv} \cdot v_z)\, z + f_v + f_v \cdot xyz\right]$$

$$= \left[f_{uu} \cdot (xy\cos(xz)) + f_{uv} \cdot xy\, e^{xyz}\right]\sin(xz) + f_u \cdot x.\cos(xz) + xze^{xyz}\left[f_{vu}\, xy\cos(xz) + f_{vv}\, xy\, e^{xyz}\right] + x e^{xyz} f_v\,(1 + xyz)$$

Q.2 (5 + 5 + 5 = **15 pts**) Let $f = f(x, y) = 5\cos(\frac{\pi}{2}x^2y)\ \ln(1 + x^2y^2)$.

(a) Find the directional derivative of f at the point $P(1, 2)$ in the direction of the vector $\vec{v} = \langle 1, -3 \rangle$.

· since $\|\vec{v}\| = \sqrt{10}$, $\vec{w} = \frac{\vec{v}}{\|\vec{v}\|} = \langle \frac{1}{\sqrt{10}}, \frac{-3}{\sqrt{10}} \rangle$ is the unit vector in direction of $\vec{v}$.

· since $f(x,y)$ is a product of trigonometric function and a logarithmic function with positive input $(1+x^2y^2>0)$, it is differentiable (at P) and hence for any direction $\vec{w}$, $D_{\vec{w}} f(P) = \nabla f(P) \cdot \vec{w}$.

$f_x(x,y) = -5\sin(\frac{\pi}{2}x^2y)(\pi xy)\ln(1+x^2y^2) + 5\cos(\frac{\pi}{2}x^2y)\frac{2xy^2}{1+x^2y^2}$

$\Rightarrow f_x(1,2) = -5\ \frac{8}{5} = -8$

$f_y(x,y) = -5\sin(\frac{\pi}{2}x^2y)\frac{\pi}{2}x^2\ln(1+x^2y^2) + 5\cos(\frac{\pi}{2}x^2y)\frac{2x^2y}{1+x^2y^2}$

$\Rightarrow f_y(1,2) = -5 \cdot \frac{4}{5} = -4$

$\therefore \nabla f(1,2) = \langle -8, -4 \rangle$.

Thus $D_{\vec{w}} f(1,2) = \nabla f(1,2) \cdot \vec{w} = \langle -8, -4 \rangle \cdot \langle \frac{1}{\sqrt{10}}, \frac{-3}{\sqrt{10}} \rangle = \frac{-8}{\sqrt{10}} + \frac{12}{\sqrt{10}}$

$= \frac{4}{\sqrt{10}}$

(b) Find the direction(s) in which the directional derivative of f at $P(1, 2)$ is 4.

Let $\vec{u} = \langle a, b \rangle$ be a unit vector ($\Rightarrow a^2+b^2=1$). Then

$D_{\vec{u}} f(1,2) = \nabla f(1,2) \cdot \vec{u} = 4$

$\Rightarrow \langle -8, -4 \rangle \cdot \langle a, b \rangle = -8a - 4b = 4 \Rightarrow -2a - b = 1$

$\Rightarrow b = -2a - 1$

$\Rightarrow a^2 + (-2a-1)^2 = 1 \Rightarrow a^2 + 4a^2 + 4a + \cancel{1} = \cancel{1}$

$\Rightarrow 5a^2 + 4a = 0 \Rightarrow a(5a+4) = 0 \Rightarrow$ either $a = 0 \Rightarrow b = -1$

$\Rightarrow \vec{u}_1 = \langle 0, -1 \rangle$

or $5a = -4 \Rightarrow a = -4/5$

$\Rightarrow b = 3/5$

$\Rightarrow \vec{u}_2 = \langle -\frac{4}{5}, \frac{3}{5} \rangle$

(c) Find the direction in which f increases most rapidly at $P(1, 2)$. What is the maximum rate of change in f at $P(1, 2)$?

$D_{\vec{u}} f(1,2) = \nabla f(1,2) \cdot \vec{u} = \|\nabla f(1,2)\| \overset{\le 1}{\underset{-1 \le}{\cos\theta}} \le \|\nabla f(1,2)\| = \underset{\vec{u}}{\mathrm{Max}}\ D_{\vec{u}} f(1,2)$

and $D_{\vec{u}} f(1,2) = \|\nabla f(1,2)\|$

if $\cos\theta = 1 \Leftrightarrow$ if $\theta = 0 \Leftrightarrow \vec{u} = \frac{\nabla f(1,2)}{\|\nabla f(1,2)\|} = \frac{\langle -8, -4 \rangle}{\sqrt{64+16}}$

$= \langle \frac{-2}{\sqrt{5}}, \frac{-1}{\sqrt{5}} \rangle$

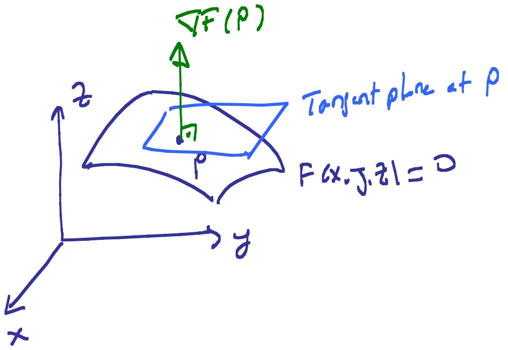
Maximum rate of change at P is $\|\nabla f(1,2)\| = \sqrt{64+16}$

$= 4\sqrt{5}$

Q.3 (5+5+5 = **15 pts**) Let $z = f(x, y)$ be the surface $\mathcal{S}$ in $\mathbb{R}^3$ containing the point $P(-2, 1, 0)$, which is defined implicitly by the equation

$$F(x, y, z) = xy^2 + 3e^{xyz} + \sin(x + 2y + z) - 1 = 0.$$

(a) Find an equation of the plane **tangent** to $\mathcal{S}$ at P.



$\nabla F(x,y,z) = \langle F_x(x,y,z), F_y(x,y,z), F_z(x,y,z) \rangle$

$F_x(x,y,z) = y^2 + 3yz\,e^{xyz} + \cos(x+2y+z)$

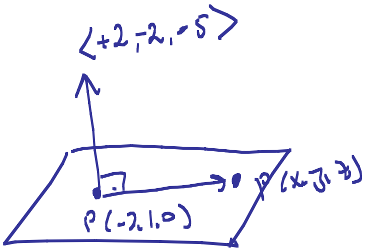
$\Rightarrow F_x(-2,1,0) = 1 + \cos 0 = 2$

$F_y(x,y,z) = 2xy + 3xz\,e^{xyz} + \cos(x+2y+z) \cdot 2$

$\Rightarrow F_y(-2,1,0) = -4 + 2\cos 0 = -2$

$F_z(x,y,z) = 3xy\,e^{xyz} + \cos(x+2y+z)$

$\Rightarrow F_z(-2,1,0) = -6 + \cos 0 = -5$



$\Rightarrow \nabla F(P) = \langle +2, -2, -5 \rangle$

$P(x,y,z)$ is on the tangent plane iff

$+2(x+2) - 2(y-1) - 5z = 0$

That is $\quad 2x - 2y - 5z = -6 \quad$ or $\quad z = \frac{2}{5}(x+2) - \frac{2}{5}(y-1)$

(b) Write down the value of z when $(x, y) = (-2, 1)$, i.e., find $z = f(-2, 1)$. Use implicit differentiation to find also the partial derivative values $z_x = f_x(-2, 1)$ and $z_y = f_y(-2, 1)$.

A point on the surface has coordinates $(x, y, z = f(x,y))$

Since $P(-2,1,0)$ is on the surface $0 = f(-2,1)$

Taking derivative of both sides of the equation $F(x,y,z) = 0$ with respect to x, we obtain

$$F_x + F_z \cdot \underbrace{\boxed{z_x}}_{= f_x} = 0$$

Hence $\quad z_x(x,y) = f_x(x,y) = \dfrac{-F_x(x,y,z)}{F_z(x,y,z)}$

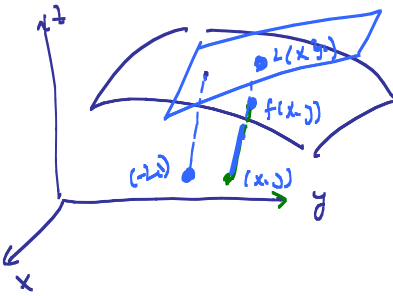
Thus $\quad f_x(-2,1) = \dfrac{-F_x(-2,1,0)}{F_z(-2,1,0)} = \dfrac{-2}{-5} = 2/5$

Similarly $\quad f_y(-2,1) = \dfrac{-F_y(-2,1,0)}{F_z(-2,1,0)} = -\dfrac{-2}{-5} = -\dfrac{2}{5}$

(c) Use a linear approximation to find $z = f(-1.95, 0.95)$ approximately.

Let $z = L(x,y)$ be the equation of the tangent plane to the surface at P.

Then $z = f(x,y) \approx L(x,y)$ is a "good approximation" for f if (x,y) is close to $(-2,1)$



Tangent plane at P $\rightarrow$ $z = \frac{2}{5}(x+2) - \frac{2}{5}(y-1)$

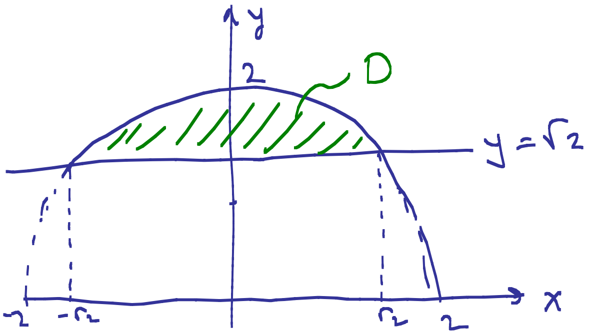
$f(x,y) \approx \frac{2}{5}(x+2) - \frac{2}{5}(y-1)$

$f(-1.95, 0.95) \approx \frac{2}{5}\, 0.05 - \frac{2}{5}(-0.05) = \frac{2}{5}\left[\frac{1}{20} + \frac{1}{20}\right]$

$= \frac{2}{50} = 0.04$

Q.4 (10 + 7 + 3 = **20 pts**) Let $\mathcal{D}$ be the region in $\mathbb{R}^2$ lying inside the circle $x^2+y^2=4$ and above the line $y=\sqrt{2}$. Let $f(x,y)$ be a continuous function on $\mathcal{D}$.

(a) Express $\displaystyle\iint_{\mathcal{D}} f(x,y)\,\mathrm{d}A$ as an iterated integral in Cartesian coordinates in **both** orders of integration $\mathrm{d}x\,\mathrm{d}y$ and $\mathrm{d}y\,\mathrm{d}x$.



$$D = \{(x,y)\in \mathbb{R}^2;\ -\sqrt{2}\le x\le \sqrt{2},\ \sqrt{2}\le y\le \sqrt{4-x^2}\}$$
$$= \{(x,y)\in \mathbb{R}^2;\ \sqrt{2}\le y\le 2,\ -\sqrt{4-y^2}\le x\le \sqrt{4-y^2}\}$$

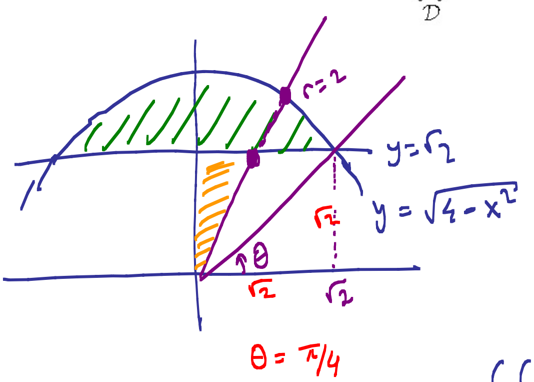
$x^2+y^2=4$ and $y=\sqrt{2}$

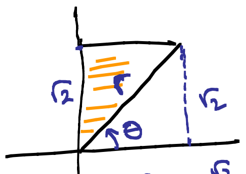
$\Rightarrow x^2 = 2 \Rightarrow x = \mp\sqrt{2}$

$$\iint_D f(x,y)\,dA = \int_{-\sqrt{2}}^{\sqrt{2}}\int_{\sqrt{2}}^{\sqrt{4-x^2}} f(x,y)\,dy\,dx$$

$$= \int_{\sqrt{2}}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} f(x,y)\,dx\,dy$$

(b) Express $\displaystyle\iint_{\mathcal{D}} f(x,y)\,\mathrm{d}A$ as an iterated integral in Polar coordinates.





$\sin\theta = \frac{\sqrt{2}}{r}$

$r = \frac{\sqrt{2}}{\sin\theta}$

$$D = \{(r,\theta) : \tfrac{\pi}{4}\le\theta\le\tfrac{3\pi}{4};\ \tfrac{\sqrt{2}}{\sin\theta} < r < 2\}$$

Thus

$$\iint_D f(x,y)\,dA = \int_{\pi/4}^{3\pi/4}\int_{\frac{\sqrt{2}}{\sin\theta}}^{2} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

(c) Express the AREA of $\mathcal{D}$ as a double integral in polar coordinates, but do **NOT** evaluate it.

$$\text{Area}(D) = \int_{\pi/4}^{3\pi/4}\int_{\frac{\sqrt{2}}{\sin\theta}}^{2} 1\cdot r\, dr\, d\theta \quad \text{where } f\equiv 1.$$

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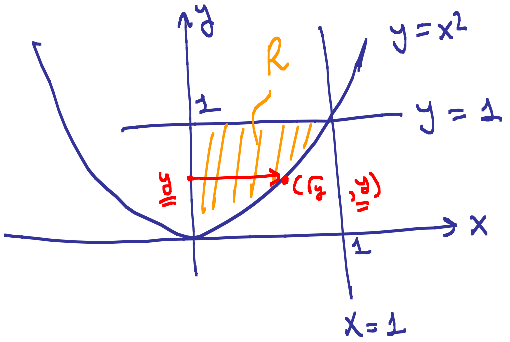
Q.5 (6 + 9 = **15 pts**) **(a)** Evaluate the iterated integral $\displaystyle\int_0^1\int_{x^2}^1 x^3\sin(y^3)\,dy\,dx$.



$$D=\{(x,y)\in\mathbb{R}^2 : 0\le x\le 1,\ x^2\le y\le 1\}$$

$$\int_0^1\int_{x^2}^1 x^3\sin(y^3)\,dy\,dx = \iint_R x^3\sin(y^3)\,dA$$

Change the order of integration

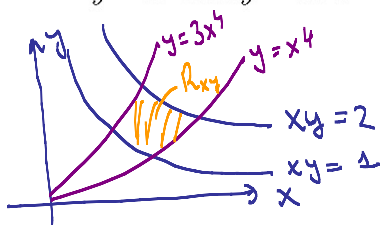
$$= \int_0^1\int_0^{\sqrt{y}} x^3\sin(y^3)\,dx\,dy$$

$$= \int_0^1 \frac{x^4}{4}\sin(y^3)\Big|_0^{\sqrt{y}}\,dy = \frac{1}{4}\int_0^1 y^2\sin(y^3)\,dy \qquad u=y^3 \Rightarrow \frac{du}{dy}=3y^2 \qquad y\in[0,1]\to u\in[0,1]$$

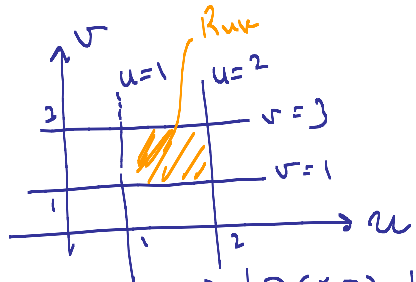
$$= \frac{1}{4}\frac{1}{3}\int_0^1 \sin u\,du = \frac{1}{12}\cdot\left(-\cos u\Big|_0^1\right)$$

$$= -\frac{1}{12}(\cos 1-1) = \frac{1}{12}(1-\cos 1)$$

(b) Using a suitable transformation (or change of variables), evaluate the double integral $\displaystyle\iint_{\mathcal{D}} x^2y^2\,dA$, where $\mathcal{D}$ is the region bounded by the curves $xy=1$, $xy=2$, $y=x^4$ and $y=3x^4$.



$$u=xy\in[1,2] \qquad v=\frac{y}{x^4}\in[1,3]$$



$$\iint_{R_{xy}} f(x,y)\,dy\,dx = \iint_{R_{uv}} f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|du\,dv =$$

$$\left(\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}} = \frac{1}{\det\begin{bmatrix}u_x & u_y\\ v_x & v_y\end{bmatrix}} = \frac{1}{\det\begin{bmatrix}y & x\\ -\frac{4y}{x^5} & \frac{1}{x^4}\end{bmatrix}} = \frac{1}{\frac{y}{x^4}+\frac{4y}{x^4}} = \frac{1}{5\frac{y}{x^4}} = \frac{1}{5v}\right)$$

$$f(x,y)=x^2y^2=u^2$$

$$= \int_1^3\int_1^2 u^2\left|\frac{1}{5v}\right|du\,dv = \frac{1}{5}\int_1^3\int_1^2 \frac{u^2}{v}\,du\,dv = \frac{1}{5}\int_1^3 \frac{u^3}{3v}\Big|_1^2\,dv$$

$$= \frac{1}{15}\int_1^3 \frac{7}{v}\,dv = \frac{7}{15}\ln|v|\Big|_1^3 = \frac{7}{15}(\ln 3-\cancel{\ln 1}^{\,0}) = \frac{7}{15}\ln 3$$

Q.6 (3 + 12 + 5 = **20 pts**) Let $\mathcal{C}$ be the CLOSED curve in the plane $\mathbb{R}^2$ defined implicitly by the equation $5x^6 + 3x^2y^2 + y^6 = 9$. Let $\mathcal{D}$ be the closed and bounded region with boundary $\mathcal{C}$, i.e., $\mathcal{D} = \{(x, y) \in \mathbb{R}^2 : 5x^6 + 3x^2y^2 + y^6 \leq 9\}$.
Let $f(x, y) = x^5 + xy^2$.

(a) Does $f(x, y)$ have an absolute maximum and absolute minimum on $\mathcal{D}$? Why?

Yes. Because $D \subseteq \mathbb{R}^2$ is a closed (since it contains it's boundary curve) and bounded region, and f being a polynomial is continuous everywhere, in particular on D. Hence by the Extreme Value Theorem f has absolute MAX and absolute min on D.

(b) Find the absolute maximum and minimum values of $f(x, y)$ on the curve $\mathcal{C}$.

We are asked to find absolute Max/min of $f(x,y)$ subject to $(x,y) \in C$

In other words: Find Max/min of $f(x,y) = x^5 + xy^2$ subject to $g(x,y) = 5x^6 + 3x^2y^2 + y^6 - 9 = 0$ — we can use Lagrange multipliers method.

Set $\nabla f = \lambda \nabla g$:

$$f_x = 5x^4 + y^2 \overset{(1)}{=} \lambda(30x^5 + 6xy^2) = \lambda g_x$$
$$f_y = 2xy \overset{(2)}{=} \lambda(6x^2y + 6y^5) = \lambda g_y$$
$$5x^6 + 3x^2y^2 + y^6 - 9 \overset{(3)}{=} 0$$

3 equations in 3 unknowns: x, y, λ.

(1) $\Rightarrow 5x^4 + y^2 = 6\lambda x(5x^4 + y^2) \Rightarrow (5x^4+y^2)(1 - 6\lambda x) = 0$

or:

- $5x^4 + y^2 = 0 \Rightarrow (x,y) = (0,0) \Rightarrow$ contradicts with Equation (3)
- $1 - 6\lambda x = 0 \Rightarrow 1 = 6\lambda x$ ($\Rightarrow x \neq 0$) $\Rightarrow \lambda = \frac{1}{6x}$

$\lambda = \frac{1}{6x}$ & (2) $\Rightarrow 2xy = \frac{1}{\cancel{6}x}(\cancel{6}x^2y + \cancel{6}y^5)$

$\rightarrow 2x^2y = x^2y + y^5 \Rightarrow x^2y - y^5 = y(x^2 - y^4) = 0$

or:

- $y = 0 \overset{(3)}{\Rightarrow} 5x^6 = 9 \Rightarrow x = \mp\left(\frac{9}{5}\right)^{1/6} \Rightarrow$

$$f\left(\left(\tfrac{9}{5}\right)^{1/6}, 0\right) = \left(\tfrac{9}{5}\right)^{5/6}$$
$$f\left(-\left(\tfrac{9}{5}\right)^{1/6}, 0\right) = -\left(\tfrac{9}{5}\right)^{5/6}$$

- $x^2 = y^4 \overset{(3)}{\Rightarrow} 5y^{12} + 3y^6 + y^6 - 9 = 0$

$5y^{12} + 4y^6 - 9 = 0$

$(5y^6 + 9)(y^6 - 1) = 0$

or:

- $y^6 = -\frac{9}{5}$ impossible
- $y^6 = 1 \Rightarrow y = \mp 1 \Rightarrow x = \mp 1 \Rightarrow$

$$f(1, \mp 1) = 2$$
$$f(-1, \mp 1) = -2$$

Since $-2 < -\left(\frac{9}{5}\right)^{5/6} < \left(\frac{9}{5}\right)^{5/6} < 2$

we have

Abs. MAX $= 2$

Abs. min $= -2$

(c) Find the absolute maximum and minimum values of $f(x, y)$ on the region $\mathcal{D}$.

Check, • boundary pts of D: from part (b); $\boxed{f(1,\mp 1) = 2}$, $\boxed{f(-1,\mp 1) = -2}$

• interior points of D: set $\nabla f = \vec{0}$ to find critical pts inside D (f has no singular pts.)

$$f_x = 5x^4 + y^2 = 0, \quad f_y = x^2 = 0 \Rightarrow (x,y) = (0,0)$$ is the only critical pt (and it is inside D)

$\boxed{f(0,0) = 0}$

Thus

Abs. max value on D is 2

Abs. min " " " is -2

M E T U
Department of Mathematics

	Calculus of Functions of Several Variables First Midterm Exam	

Code : MATH 120 Acad. Year: 2014-2015 Semester : Spring Coord. : Muhiddin Uğuz Date : 04.04.2015 Time : 9.30 Duration : 135 minutes	Last Name : Name : Student No : Department: Section No : Signature :
	6 Questions on 6 Pages Total 100 Points

1	2	3	4	5	6	SHOW YOUR WORK	

Q.1 (5+2+5+3 = **15 pts**) Consider the series $\sum_{n=1}^{\infty} a_n$ whose n-th partial sum S_n is given by

$$S_n = \frac{2n^3+4n+1}{n^3+8n^2+5n+2}\,.$$

(a) Does the series $\sum_{n=1}^{\infty} a_n$ converge? If it is convergent find its SUM.

Recall that $\sum_{n=1}^{\infty} a_n \overset{\text{defn}}{=} \lim_{n\to\infty} S_n$

since $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{\cancel{n^3}\,(2+\frac{4}{n^2}+\frac{1}{n^3})}{\cancel{n^3}\,(1+\frac{8}{n}+\frac{5}{n^2}+\frac{2}{n^3})} = 2$ we have $\sum_{n=1}^{\infty} a_n = 2$ (convergent)

(b) Is the sequence $\{a_n\}$ convergent?

Since $\sum a_n$ is convergent by part (a), using n^{th} term test we can say $\lim_{n\to\infty} a_n = 0$. Hence the sequence $\{a_n\}$ is convergent

(c) Let $b_n = \dfrac{e^{a_n}-1}{2a_n}$. Is the sequence $\{b_n\}$ convergent?

$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{e^{a_n}-1}{2a_n} = \lim_{t\to 0} \frac{e^t-1}{2t} = \frac{1}{2}$ Hence $\{b_n\}$ is a convergent sequence

Let $t = a_n$
then as $n\to\infty$, $t\to 0$

(d) Let, again, $b_n = \dfrac{e^{a_n}-1}{2a_n}$. Is the series $\sum_{n=1}^{\infty} b_n$ convergent?

$\lim_{n\to\infty} b_n = \frac{1}{2} \neq 0$, hence by n^{th} term test, $\sum b_n$ is divergent

Q.2 ($4 \times 5 =$ **20 pts**) Determine whether the series are convergent or divergent:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{e^{3/n}}{n^2}$ Let $a_n = \frac{e^{3/n}}{n^2} > 0 \ \forall n = 1, 2 ..$

and $b_n = \frac{1}{n^2} > 0 \ \forall n = 1, 2 ..$

since $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{e^{3/n}}{n^2} \cdot \frac{n^2}{1} = \lim_{n\to\infty} e^{\frac{3}{n}} = 1$ = a nonzero finite number,

by limit comparison test either both $\sum a_n$ & $\sum b_n$ converge, or both diverge

Since $\sum b_n = \sum \frac{1}{n^2}$ is convergent by p-test (or by integral test),

we have $\sum a_n$ is also convergent

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n! + \frac{1}{120}}{(n+1)!} = \sum a_n$ where $a_n > 0 \ \forall n = 1, 2 ..$

Since $0 \leq \frac{1}{n+1} = \frac{n!}{(n+1)!} \leq \frac{n! + \frac{1}{120}}{(n+1)!}$

and since $\sum \frac{1}{n+1}$ is divergent harmonic series,

by comparison test we have $\sum a_n$ is divergent

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{2^{2n+3}\sin(1+n^2)}{5^n} = \sum_{n=0}^{\infty} a_n$

$0 \leq |a_n| \leq \frac{2^{2n+3}}{5^n} = 8 \left(\frac{4}{5}\right)^n \quad \forall n = 0, 1, ...$

since $\sum 8 \left(\frac{4}{5}\right)^n = 8 \sum \left(\frac{4}{5}\right)^n = 8 \sum r^n$ is convergent geometric series ($|r| = \frac{4}{5} < 1$),

by comparison test $\sum |a_n|$ is convergent,

and hence $\sum a_n$ is absolutely convergent.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^3 2^{5n}}{(3n)!} = \sum a_n \quad ; \ a_n > 0 \ \forall n = 1, 2 ...$

Let's use Ratio Test:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^3 \cancel{(n!)^3}\, \cancel{2^{5n}} \cdot 2^5}{(3n+3)(3n+2)(3n+1)\cancel{(3n)!}} \cdot \frac{\cancel{(3n)!}}{\cancel{(n!)^3}\, \cancel{2^{5n}}}$$

$$= 2^5 \lim_{n\to\infty} \frac{(n+1)^3}{(3n+3)(3n+2)(3n+1)} = \frac{2^5}{27} = \frac{32}{27} > 1$$

Hence by Ratio Test $\sum a_n$ is divergent.

Q.3 (10+5 = **15 pts**) Let f be the function defined by $f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n n}{5^n(n^2+3)} x^{2n+1}$

(a) What is the largest possible domain of f?

(Largest possible domain of f) = All x's for which the series converges
= interval of convergence of the power series = $\underline{\underline{I}}$

For testing absolute convergence, apply ratio test :

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)}{5^{\cancel{n+1}}(n^2+2n+1+3)} \cancel{|x|^{2n+3}}^{\,|x|^2} \cdot \frac{\cancel{5^n}\,(n^2+3)}{n\ \cancel{|x|^{2n+1}}}$$

$$= \frac{1}{5} x^2 \lim_{n\to\infty} \frac{(n+1)(n^2+3)}{n^3+2n^2+4n} = \frac{1}{5}x^2$$

Given Power Series is (absolutely) convergent if $\frac{1}{5}x^2 < 1$
diverges if $\frac{1}{5}x^2 > 1$

$\Rightarrow$ Radius of convergence $R = \sqrt{5}$

Check end points:

$x = \sqrt{5} \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n n}{\cancel{5^n}(n^2+3)} \cancel{5^n}\sqrt{5} = \sqrt{5} \sum (-1)^n \frac{n}{n^2+3}$ is an Alternating series
$= \sqrt{5} \sum (-1)^n a_n$ where $a_n > 0$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n^2+3} = 0$ ✓ Moreover if $f(x) = \frac{x}{x^2+3}$, then $f'(x) = \frac{3-x^2}{(x^2+3)^2} < 0$ if $x > \sqrt{3}$
i.e. $f(x)$ and hence a_n is decreasing
Thus $\sum (-1)^n a_n$ is conv. by AST.

$x = -\sqrt{5} \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n n}{5^n(n^2+3)} (-5^n\sqrt{5}) = -\sqrt{5} \sum (-1)^n \frac{n}{n^2+3}$ is convergent alternating series as above

Hence $I = [-\sqrt{5}, \sqrt{5}]$

(b) Let g be the function defined by $g(x) = f(x^2)$. Write down the power series expansion of g, and then evaluate its *42nd* derivative at $x = 0$, i.e. find $g^{(42)}(0)$.

$$g(x) = f(x^2) = \sum_{n=1}^{\infty} \frac{(-1)^n n}{5^n(n^2+3)} x^{4n+2} = \sum_{n=1}^{\infty} \frac{g^{(n)}(0)}{n!} x^n$$

$\Downarrow$

$\frac{g^{(42)}(0)}{42!}$ is coefficient of x^{42}

$4n+2 = 42 \Rightarrow 4n = 40$
$\Rightarrow n = 10$

$$\Rightarrow \frac{g^{(42)}(0)}{42!} = \frac{(-1)^{10}\cdot 10}{5^{10}\cdot 103} \Rightarrow g^{(42)}(0) = \frac{42!\cdot 10}{5^{10}\cdot 103}$$

Q.4 ($3 \times 5 =$ **15 pts**)

Consider the piecewise defined function $f(x) = \begin{cases} \frac{1}{x}\sin(x^2) & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$ which is continuous and infinitely many times differentiable for all x.

(a) Use the Maclaurin series of $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$ to find the Maclaurin series of $f(x)$.

$$\sin(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{4n+2}$$

- $x \neq 0 \Rightarrow \frac{\sin(x^2)}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{4n+1} = f(x) \quad \forall x \in \mathbb{R}\setminus\{0\}$
- $x = 0 \Rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{(2n+1)!} = x - \frac{x^5}{6} + \cdots \Big|_{x=0} = 0 = f(0)$

Hence $$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{4n+1} \quad \forall x \in \mathbb{R}$$

(b) Use the series of $f(x)$ found in Part **(a)** to evaluate the integral $I = \int_0^1 f(x)\mathrm{d}x$ as an infinite series, and show that it is convergent.

$$\int_0^x f(t)\,dt = \int_0^x \left(\sum_{n=0}^{\infty} \frac{(-1)^n t^{4n+1}}{(2n+1)!} \right) dt = \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(4n+2)(2n+1)!}}_{\text{Radius of convergence does not change after integration}} \quad \forall x \in \mathbb{R}$$

$$x = 1 \Rightarrow \int_0^1 f(x)\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+2)(2n+1)!}$$

Since power series obtained for $f(x)$ is convergent $\forall x$, so is it's integral. Hence it's integral $\int_0^1 f(x)\,dx = \sum \frac{(-1)^n}{(4n+2)(2n+1)!}$ is also convergent.

(c) Use the series in Part **(b)** to find an approximate value of the integral I with an error, absolute value of which is less than 10^{-4}.

$I = \sum_{n=0}^{\infty} (-1)^n b_n$ where
- $b_n = \frac{1}{(4n+2)(2n+1)!} > 0$
- $b_n \searrow$
- $\lim_{n\to\infty} b_n = 0$

Hence by Alternating Series Error Bound we have

$$\underbrace{|I - S_n|}_{S_n = b_0 - b_1 + b_2 \cdots \mp b_n} < b_{n+1} = \frac{1}{(4n+6)(2n+3)!} < \frac{1}{10.000} \quad \forall n \geq 2. \quad \text{Thus}$$

$$I \approx S_2 = \frac{1}{2} - \frac{1}{6 \cdot 3!} + \frac{1}{10 \cdot 5!} = \frac{1}{2} - \frac{1}{36} + \frac{1}{1200}$$

NAME : **SURNAME :**

Q.5 (3 × 5 = **15 pts**) Let $f(x,y) = \begin{cases} \dfrac{x^2y+y^3}{x^6+2y^2} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases}$

(a) Show that the limit of $f(x,y)$ as $(x,y) \to (0,0)$ does NOT exist.

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=mx^2}} f(x,y) = \lim_{\substack{x\to 0\\ (x\neq 0)}} f(x, mx^2) = \lim_{x\to 0} \frac{x^2 m x^2 + m^3 x^6}{x^6 + 2m^2x^4}$$

$$= \lim_{x\to 0} \frac{m + m^3x^2}{x^2+2m^2} = \frac{m}{2m^2} = \frac{1}{2m}$$

depends on m. Thus $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist

(b) Calculate $f_x(0,0)$ and $f_y(0,0)$, if they exist.

$$f_x(0,0) = \lim_{t\to 0} \frac{f(t,0) - \overbrace{f(0,0)}^{0}}{t} = \lim_{t\to 0} \frac{\frac{0}{t^6}}{t} = \lim_{t\to 0} 0 = \underline{\underline{0}}$$

$$f_y(0,0) = \lim_{t\to 0} \frac{f(0,t) - f(0,0)}{t} = \lim_{t\to 0} \frac{\frac{t^3}{2t^2}}{t} = \frac{1}{2} \,//$$

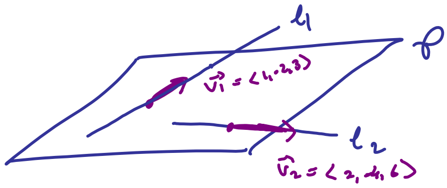
(c) Write $f_y(x,y)$ as a piecewise defined function like $f(x,y)$.

$$f_y(x,y) = \begin{cases} \dfrac{(x^2+3y^2)(x^6+2y^2) - 4y(x^2y+y^3)}{(x^6+2y^2)^2} & \text{if } (x,y) \neq (0,0) \\ 1/2 & \text{if } (x,y) = (0,0) \end{cases}$$

Q.6 (6+6+6+2 = **20 pts**) For $t \in \mathbb{R}$, let ℓ_1 and ℓ_2 be two lines given by

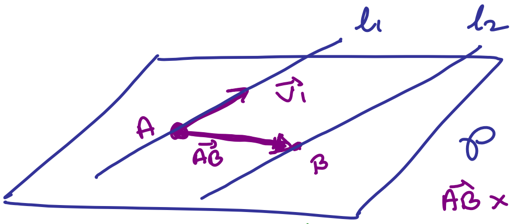
$$\ell_1 : \langle x, y, z\rangle = \langle 1+t\,, -2t\,,\, 2+3t\rangle \quad\text{and}\quad \ell_2 : \langle x, y, z\rangle = \langle 2+2t\,, 2-4t\,,\, 3+6t\rangle.$$

(a) Determine an equation of the plane $\mathcal{P}_1$, which contains both ℓ_1 and ℓ_2.



$\vec v_2 = 2\vec v_1$ hence $\vec v_1 \parallel \vec v_2 \Rightarrow \ell_1 \parallel \ell_2$

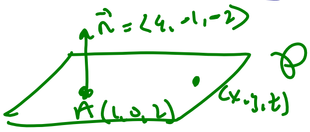
$\Rightarrow$ There is a unique plane $\mathcal{P}$ containing the parallel lines ℓ_1 & ℓ_2 $(\ell_1 \neq \ell_2)$



$t=0 \Rightarrow \left.\begin{array}{l} A=(1,0,2)\in\ell_1 \\ B=(2,2,3)\in\ell_2 \end{array}\right\}$ $\vec{AB} = \langle 1,2,1\rangle$ is parallel to $\mathcal{P}$

Also $\vec v_1$ is parallel to $\mathcal{P}$. Hence normal vector $\vec n$ of $\mathcal{P}$ is parallel to $\vec{AB}\times\vec v_1$

$\vec{AB}\times\vec v_1 = \begin{vmatrix} i & j & k \\ 1 & 2 & 1 \\ 1 & -2 & 3 \end{vmatrix} = \langle 8,-2,-4\rangle \Rightarrow$ we can take $\vec n = \langle 4,-1,-2\rangle$

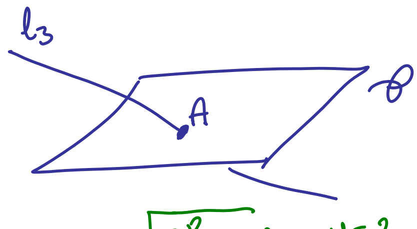


$P(x,y,z)\in\mathcal{P}$ iff $4(x-1)-1(y-0)-2(z-2)=0$

i.e. $\boxed{4x-y-2z=0}$

(b) Find the intersection point, say A, of the plane $\mathcal{P}_2 :\ x+y-z=2$ and the line ℓ_3 given by the symmetric equations $\ell_3 : \dfrac{x+3}{4} = \dfrac{z+1}{2}$ and $y=2$.

Parametric equation of ℓ_3 : $\begin{array}{l} x=-3+4t \\ y=2 \\ z=-1+2t \end{array}$



$A(x,y,z)\in\ell_3 \Longrightarrow A(-3+4t,\ 2,\ -1+2t)$

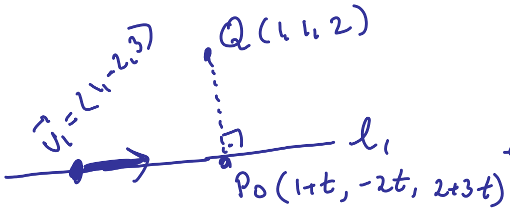
$A(x,y,z)\in\mathcal{P} \Longrightarrow x+y-z=2$

Then $-3+4t+\cancel{2}+1-2t = \cancel{2} \Longrightarrow 2t=2 \Longrightarrow t=1$

$\Longrightarrow A=(1,2,1) = \mathcal{P}_2\cap\ell_3$

OR since $y=2$, $x-z=0$, $x=z$
and $4z+4 = 2x+6$
$4x+4 = 2x+6$
$2x=2$
$x=1=z$
$A(x,y,z) = (1,2,1)$

(c) Find the point P_0 on line ℓ_1, which is <u>closest</u> to the point $Q=(1,1,2)$.



P_0 is the point on ℓ_1 that is closest to Q

$\Longleftrightarrow \vec{P_0Q}\perp\vec v_1 \Longleftrightarrow \vec{P_0Q}\cdot\vec v_1 = 0$

$\Longleftrightarrow \langle -t, 1+2t, -3t\rangle\cdot\langle 1,-2,3\rangle = 0$

$-t-2-4t-9t=0 \Longrightarrow -14t=2$

$\Longrightarrow t=-1/7$

$\Longrightarrow P_0\left(\frac{7}{7}-\frac{1}{7},\ \frac{2}{7},\ \frac{14}{7}-\frac{3}{7}\right) = \left(\frac{6}{7},\frac{2}{7},\frac{11}{7}\right)$

(d) Find the <u>distance</u> between point Q and line ℓ_1.

From part (d), $d = |\vec{P_0Q}| = \sqrt{\left(\frac{1}{7}\right)^2+\left(1-\frac{2}{7}\right)^2+\left(\frac{3}{7}\right)^2} = \sqrt{\frac{1+25+9}{49}}$

$= \frac{\sqrt{35}}{7}$