M E T U
Department of Mathematics

	Calculus of Functions of Several Variables First Midterm Exam	
Code : MATH 120 Acad. Year : 2015-2016 Semester : Spring Coord. : Muhiddin Uğuz Date : 09.04.2016 Time : 9.30 Duration : 120 minutes	Last Name : Name : Student No. : Department : Section : Signature : **6** Questions on **6** Pages Total **100** Points	

1	2	3	4	5	6	SHOW YOUR WORK	

1. (6+6=12 pts) Suppose that $a_n > 0$ for every $n \geq 1$. If the series $\sum_{n=1}^{\infty} a_n$ is convergent, determine whether the following series are convergent or divergent. Explain.

(a) $\sum_{n=1}^{\infty} e^{a_n}$

Since $\sum_{n=1}^{\infty} a_n$ is convergent, by n^{th} term test $\lim_{n\to\infty} a_n = 0$

Then $\lim_{n\to\infty} e^{a_n} \underset{e^x \text{ is continuous}}{=} e^{\lim_{n\to\infty} a_n} = e^0 = 1 \neq 0.$

Thus by n^{th} term test $\sum_{n=1}^{\infty} e^{a_n}$ is divergent

(b) $\sum_{n=1}^{\infty} \frac{a_n}{n}$

For $n \geq 1$ we have $a_n > 0$ and hence $0 < \frac{a_n}{n} \leq a_n$

Since $\sum a_n$ is convergent,

$\sum_{n=1}^{\infty} \frac{a_n}{n}$ is convergent by Comparison Test

(or/ $\left.\begin{array}{l} \frac{a_n}{n} > 0 \\ a_n > 0 \end{array}\right\}$ $\lim_{n\to\infty} \frac{a_n/n}{a_n} = \lim_{n\to\infty} \frac{1}{n} = 0$ and $\sum a_n$ is conv.

Thus by Limit Comp. Test, $\sum \frac{a_n}{n}$ is also conv.)

2. (16 pts) Find the radius of convergence and interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(2x+3)^n}{3^{2n+1}\sqrt{n}}$$

Apply ratio test: (or you may apply root test...)

$$\lim_{n\to\infty} \frac{|2x+3|^{n+1}}{3^{2(n+1)+1}\sqrt{n+1}} \cdot \frac{3^{2n+1}\sqrt{n}}{|2x+3|^n} = \frac{|2x+3|}{9} \lim_{n\to\infty} \sqrt{\frac{n}{n+1}} = \frac{|2x+3|}{9}$$

If $\frac{|2x+3|}{9} < 1 \iff \left|x+\frac{3}{2}\right| < \frac{9}{2} \Rightarrow$ Radius of convergence $R = \frac{9}{2}$

$\iff -\frac{9}{2} < x + \frac{3}{2} < \frac{9}{2}$

$\iff -6 < x < 3$ (Given power series is absolutely convergent on $(-6,3)$, is divergent on $(-\infty,-6) \cup (3,+\infty)$).

Check end points:

$x=-6 \Rightarrow \sum_{n=1}^{\infty} \frac{(2x+3)^n}{3^{2n+1}\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n 9^n}{9^n \cdot 3\sqrt{n}} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{3\sqrt{n}} = \sum_{n=1}^{\infty} (-1)^n a_n$

$a_n = \frac{1}{3\sqrt{n}} \Rightarrow$ i) $a_n > 0$ for $n \geq 1$

ii) $a_{n+1} = \frac{1}{3\sqrt{n+1}} < \frac{1}{3\sqrt{n}} = a_n \Rightarrow a_n$ is decreasing

iii) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{3\sqrt{n}} = 0$

By Alternating series test $\sum (-1)^n a_n$ is convergent

$x=3 \Rightarrow \sum_{n=1}^{\infty} \frac{(2x+3)^n}{3^{2n+1}\sqrt{n}} = \sum_{n=1}^{\infty} \frac{9^n}{9^n 3\sqrt{n}} = \frac{1}{3}\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ is divergent by p-test

Therefore the interval of convergence $I = [-6, 3)$

3. (6+6+6=18 pts)

(a) Is the series $\displaystyle\sum_{n=3}^{\infty} \frac{2n+1}{\sqrt[3]{n}+2n^2+1}$ convergent or divergent?

let $a_n = \dfrac{2n+1}{\sqrt[3]{n}+2n^2+1}$. Then $a_n > 0$. Let $b_n = \dfrac{1}{n} > 0$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n^2+n}{\sqrt[3]{n}+2n^2+1} = \lim_{n\to\infty} \frac{\cancel{n^2}\left(2+\frac{1}{n}\right)}{\cancel{n^2}\left(\frac{1}{n^{5/3}}+2+\frac{1}{n^2}\right)} = \frac{2}{2} = 1$$

Since 1 is nonzero finite number, by limit comparison test either both $\sum a_n$ & $\sum b_n$ converge or both diverge.

$\sum b_n = \sum \frac{1}{n}$ is divergent harmonic series, and hence $\sum a_n$ is divergent

(b) Is the series $\displaystyle\sum_{n=1}^{\infty} \frac{n\pi^n}{2^{2n+1}} \sin n$ convergent or divergent?

$$\sum \frac{n\pi^n}{2^{2n+1}} \sin n = \sum a_n$$

$0 \le |a_n| \le \underbrace{\boxed{\dfrac{n.\pi^n}{2^{2n+1}}}}_{b_n}$. Let's use comparison test to check if $\sum |a_n|$ is convergent (and hence $\sum a_n$ is absolutely conv.)

Note that $b_n > 0$ $\forall n = 1, 2 \ldots$

Moreover $\displaystyle\lim_{n\to\infty} \frac{b_{n+1}}{b_n} = \lim_{n\to\infty} \frac{(n+1)\cancel{\pi^n}.\pi}{\cancel{2^{2n+1}}.2^2} \cdot \frac{\cancel{2^{2n+1}}}{n.\cancel{\pi^n}} = \frac{\pi}{4}\lim_{n\to\infty}\frac{n+1}{n} = \frac{\pi}{4} < 1$

Thus by ratio test $\sum b_n$ is convergent.

$\Rightarrow \sum |a_n|$ is convergent by comparison test. $\Rightarrow \sum a_n$ is (absolutely) convergent.

(c) Find the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{4}{n^2+2n} = \sum \frac{4}{n(n+2)} = \sum\left(\frac{A}{n}+\frac{B}{n+2}\right)$

$4 = A(n+2) + Bn$
$= (A+B)n + 2A$
$A+B=0$, $2A=4$
$A=2$, $B=-2$

$$= \sum_{n=1}^{\infty}\left(\frac{2}{n} - \frac{2}{n+2}\right)$$

$$= \sum_{n=1}^{\infty}(b_n - b_{n+2}) \text{ where } b_n = \frac{2}{n}$$

$$= \lim_{n\to\infty} S_n$$

$$S_n = (b_1 - \cancel{b_3}) + (b_2 - \cancel{b_4}) + (\cancel{b_3} - \cancel{b_5}) + \cdots + (\cancel{b_{n-1}} - b_{n+1}) + (\cancel{b_n} - b_{n+2})$$

$$= b_1 + b_2 - b_{n+1} - b_{n+2} = \frac{2}{1} + \frac{2}{2} - \frac{2}{n+1} - \frac{2}{n+2}$$

$$\Rightarrow \lim_{n\to\infty} S_n = 2+1 = 3 = \sum_{n=1}^{\infty} \frac{4}{n^2+2n} \text{ convergent}$$

$\Big($ or $\displaystyle x\ln|1-x| = -\sum_{n=1}^{\infty} \frac{x^{n+1}}{n}$ $\forall\, |x|<1$ $\displaystyle\Rightarrow -4\int_0^1 x\ln|1-x|\,dx = +4\sum_{n=1}^{\infty}\int_0^1 \frac{x^{n+1}}{n}\,dx = +4\sum_{n=1}^{\infty} \frac{x^{n+2}}{n(n+2)}\Big|_0^1 = \sum_{n=1}^{\infty}\frac{4}{n^2+2n}$ $\Big)$

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4.(4+6+8=18 pts) Let $f(x) = x\sin(x^3)$

(a) Find a power series representation of $f(x)$. (Hint: $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ for all $x \in \mathbb{R}$).

$$\sin(x^3) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^3)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+3}}{(2n+1)!}$$

$$\Rightarrow f(x) = x\sin(x^3) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+4}}{(2n+1)!} \quad \forall x \in \mathbb{R}.$$

(b) Find $f^{(40)}(0)$.

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \Rightarrow (\text{coefficient of } x^n) = a_n = \frac{f^{(n)}(0)}{n!}$$

To find coefficient of x^{40} set $6n+4 = 40 \Rightarrow n = 6$

$$n = 6 \Rightarrow (\text{coefficient of } x^{40}) = a_{40} = \frac{(-1)^6}{13!} = \frac{1}{13!} = \frac{f^{(40)}(0)}{40!}$$

$$\Rightarrow f^{(40)}(0) = \frac{40!}{13!}$$

(c) Approximate $\int_0^1 x\sin(x^3)\,dx$ with error less than 10^{-4}. Write down your approximate value, do not simplify.

$$\int_0^x t\sin(t^3)\,dt = \int_0^x \left(\sum_{n=0}^{\infty} \frac{(-1)^n t^{6n+4}}{(2n+1)!} \right) dt \underset{\uparrow}{=} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{1}{(6n+5)} x^{6n+5} \quad \forall x$$

term by term integration

$$x = 1 \Rightarrow \int_0^1 t\sin(t^3)\,dt = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!\,(6n+5)} = \sum_{n=0}^{\infty} (-1)^n a_n$$

i) $a_n > 0$ ii) $a_{n+1} < a_n$ so $a_n \searrow$ iii) $\lim_{n\to\infty} a_n = 0$

Hence $\sum_{n=0}^{\infty} (-1)^n a_n$ is convergent by Alternating Series Test

Moreover by the error bound formula for convergent alternating series, we have $\sum_{n=0}^{\infty} (-1)^n a_n = S \approx S_k \Rightarrow \text{error} = e_k = |S - S_k| < a_{k+1}$

k	$a_{k+1} = \frac{1}{(6k+11)(2k+3)!}$
0	$\frac{1}{11 \cdot 6} > 10^{-4}$
1	$\frac{1}{17} \cdot \frac{1}{5!} = \frac{1}{17} \cdot \frac{1}{120} > 10^{-4}$
2	$\frac{1}{23} \cdot \frac{1}{7!} = \frac{1}{23} \cdot \frac{1}{5040} < 10^{-4}$

$k = 2$: $S \approx S_2$ gives error $e_2 = |S - S_2| < 10^{-4}$

Thus

$$\int_0^1 x\sin(x^3)\,dx \approx a_0 - a_1 + a_2 = \frac{1}{5} - \frac{1}{11 \cdot 6} + \frac{1}{17 \cdot 120}$$

5. (6+6+6=18 pts)

Let K be the line given by $\dfrac{x+1}{2} = \dfrac{y-3}{4} = \dfrac{z+5}{7}$ and

L be the line given by $\begin{aligned} x &= -2+t \\ y &= -4+3t \\ z &= 14-t \end{aligned}$, $t \in \mathbb{R}$

(a) Show that K and L intersect by finding their intersection point.

$(x,y,z) \in K \Rightarrow \left[\begin{array}{l} x = -1+2s \\ y = 3+4s \\ z = -5+7s \\ s \in \mathbb{R} \end{array}\right]$

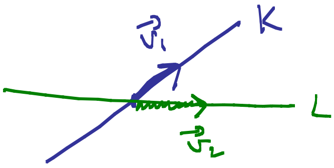
$(x,y,z) \in K \cap L \Rightarrow \left.\begin{array}{l} -1+2s = -2+t \\ 3+4s = -4+3t \\ -5+7s = 14-t \end{array}\right\} \Rightarrow \begin{array}{l} \left.\begin{array}{l} 2s-t=-1 \\ 4s-3t=-7 \end{array}\right\} t=5,\ s=2 \\ 7s+t=19 \ (*) \end{array}$

check if $(t,s)=(5,2)$ satisfies $(*)$:
$7s+t = 7\cdot 2+5 = 19$ ✓

$s=2 \Rightarrow$
$(x,y,z) = (3,11,9)$ is the point of intersection of K & L

so $\emptyset \neq K \cap L = P(3,11,9)$

(b) Find the angle $\theta \in [0, \frac{\pi}{2}]$ between K and L

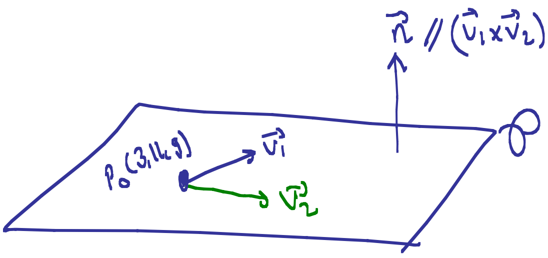


$\left.\begin{array}{l} \vec{v}_1 = \langle 2,4,7 \rangle \\ \vec{v}_2 = \langle 1,3,-1 \rangle \end{array}\right\}$ are direction vectors of K & L

$\vec{v}_1 \cdot \vec{v}_2 = \|\vec{v}_1\| \, \|\vec{v}_2\| \cos\theta = 2+12-7 = 7 = \sqrt{4+16+49} \cdot \sqrt{1+9+1} \cos\theta$

$7 = \sqrt{69} \cdot \sqrt{11} \cos\theta \Rightarrow \theta = \arccos\left(\dfrac{7}{\sqrt{759}}\right) \in [0, \pi/2]$ since $\cos\theta > 0$.

(c) Find an equation of the plane that contains both K and L



$\vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 4 & 7 \\ 1 & 3 & -1 \end{vmatrix} = \langle -25, 9, 2 \rangle$

$(x,y,z) \in \mathcal{P} \Leftrightarrow \langle -25, 9, 2 \rangle \cdot \langle x-3, y-11, z-9 \rangle = 0$

$\Leftrightarrow -25x + 75 + 9y - 99 + 2z - 18 = 0$

$\Leftrightarrow -25x + 9y + 2z = 42$

6.(6+6+6=18 pts) Consider the function

$$f(x,y) = \begin{cases} \dfrac{x^3+y^3}{2x^2+y^3} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

(a) Is f continuous at the point $(0,0)$?

Assume that $\lim_{(x,y)\to(0,0)} f(x,y)$ exists and equal to L. Then

$$L = \lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=mx}} f(x,y) = \lim_{\substack{x\to 0\\ (x\neq 0)}} f(x,mx) = \lim_{x\to 0} \frac{\cancel{x^2}\,x(1+m^3)}{\cancel{x^2}(2+m^3x)} = \frac{0}{2} = 0 \quad \forall m \in \mathbb{R}$$

But if $(x,y) \to (0,0)$ along y axis $(m\to\infty)$, that is if $x=0$ & $y\to 0$

$$L = \lim_{y\to 0} f(0,y) = \lim_{y\to 0} \frac{y^3}{y^3} = 1 \quad \text{contradicting } L=0.$$

Thus such L, that is limit does not exist and therefore $f(x,y)$ is NOT continuous at $(0,0)$.

(or simply $\lim_{y\to 0} f(0,y) = \cdots = 1 \neq \underset{\overset{\shortparallel}{0}}{f(0,0)} \Rightarrow f$ is not cont. at $(0,0)$)

(b) Find $f_1(0,0)$ if it exists.

Consider $\displaystyle \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{h^3}{2h^2} - 0}{h} = \lim_{h\to 0} \frac{h^3}{2h^3} = \frac{1}{2}$ exists.

$\therefore \; f_1(0,0) = \frac{1}{2}$

(c) Find $f_1(x,y)$ for $(x,y) \neq (0,0)$. Do not simplify the result.

if $(x,y) \neq (0,0)$ and $(x,y) \in \mathrm{Dom}(f)$, we have

$$f_1(x,y) = \frac{\partial}{\partial x} f(x,y) = \frac{\partial}{\partial x}\left(\frac{x^3+y^3}{2x^2+y^3}\right)$$

$$= \frac{3x^2(2x^2+y^3) - 4x(x^3+y^3)}{(2x^2+y^3)^2}$$

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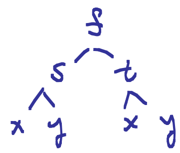
	Calculus of Functions of Several Variables Second Midterm Exam	

Code : MATH 120 Acad. Year : 2015-2016 Semester : Spring Coord. : Muhiddin Uğuz Date : 14.05.2016 Time : 9.30 Duration : 130 minutes	Last Name : Name : Student No. : Department : Section : Signature : **7** Questions on **6** Pages Total **100** Points

1	2	3	4	5	6	7	SHOW YOUR WORK	

1. (12 pts) Let $u = xf\left(\frac{y}{x^2}, \frac{z}{x^3}\right)$ where f is a function which has continuous partial derivatives. If $f(2,3) = 4$, $f_1(2,3) = 5$ and $f_2(2,3) = 6$, find $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, and $\frac{\partial u}{\partial z}$ at the point $(x,y,z) = (1,2,3)$.

Let $s = y/x^2$, $t = z/x^3$. Then $f = f(s,t)$



- $$\frac{\partial u}{\partial x} = f(s,t) + x\left[f_1(s,t)\frac{\partial s}{\partial x} + f_2(s,t)\frac{\partial t}{\partial x}\right]$$
$$= f(s,t) + x\left[f_1(s,t)\left(\frac{-2y}{x^3}\right) + f_2(s,t)\left(\frac{-3z}{x^4}\right)\right]$$
$(x,y,z) = (1,2,3) \Rightarrow (s,t) = (2,3)$

Hence $$\left.\frac{\partial u}{\partial x}\right|_{(x,y,z)=(1,2,3)} = f(2,3) + 1\left[f_1(2,3)(-4) + f_2(2,3)(-9)\right]$$
$$= 4 + (5(-4) + 6(-9)) = -70$$

- $$\frac{\partial u}{\partial y} = x\left[f_1(s,t)\frac{\partial s}{\partial y} + f_2(s,t)\underbrace{\frac{\partial t}{\partial y}}_{=0}\right] = x\left[f_1(s,t)\frac{1}{x^2}\right]$$

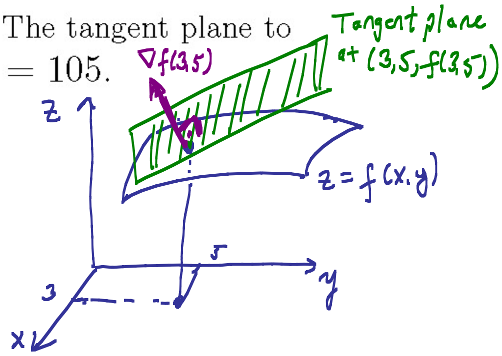
Hence $$\left.\frac{\partial u}{\partial y}\right|_{(x,y,z)=(1,2,3)} = 1\, f_1(2,3) = 5$$

- $$\frac{\partial u}{\partial z} = x\left[f_1(s,t)\underbrace{\frac{\partial s}{\partial z}}_{=0} + f_2(s,t)\frac{\partial t}{\partial z}\right] = x\left[f_2(s,t)\frac{1}{x^3}\right]$$

Hence $$\left.\frac{\partial u}{\partial z}\right|_{(x,y,z)=(1,2,3)} = \frac{f_2(2,3)}{1^2} = 6$$

2. (5+5+5=15 pts)

Let $f(x,y)$ be a function which has continuous partial derivatives. The tangent plane to the graph of $f(x,y)$ at the point $(3,5)$ is given by $7x-5y+3z=105$.



(a) Find $\nabla f(3,5)$

$$\nabla f(3,5) = \langle f_x(3,5),\ f_y(3,5) \rangle$$

Equation of the tangent plane to the surface $z=f(x,y)$ at the point $(3,5,f(3,5))$ is

$$z = f(3,5) + f_x(3,5)(x-3) + f_y(3,5)(y-5)$$ is given as $7x-5y+3z=105$, that is $z = 35 - \frac{7}{3}x + \frac{5}{3}y$

Comparing the coefficients of x & y, we obtain;

$$\left.\begin{array}{l} f_x(3,5) = -7/3 \\ f_y(3,5) = 5/3 \end{array}\right\} \quad \nabla f(3,5) = \left\langle -\frac{7}{3}, \frac{5}{3} \right\rangle$$

(b) Let C be the curve on the $xy-$plane given by $\vec{r}(t) = \langle 3t^2, 5t^3 \rangle, t \geq 0$.Find a tangent vector $\vec{v}$ to C at the point $(3,5)$ and write a vector equation of the tangent line to C at the point $(3,5)$.

$$(3t^2, 5t^3) = (3,5) \Rightarrow t^2 = 1 \ \& \ t^3 = 1 \Rightarrow t = 1$$

$\vec{v} = \vec{r}\,'(1)$ where $\vec{r}\,'(t) = \langle 6t, 15t^2 \rangle$. Thus $\vec{v} = \langle 6, 15 \rangle$

Vector equation of the tangent line to C at $(3,5)$ is:

$$L: \ \langle 3,5 \rangle + t\vec{v} \ ; t \in \mathbb{R};$$

$$\langle 3,5 \rangle + t\langle 6,15 \rangle \ ; t \in \mathbb{R}$$

(c) Find the directional derivative of $f(x,y)$ at the point $(3,5)$ in the direction of the vector $\vec{v}$ found in part (b).

$\vec{v} = \langle 6, 15 \rangle \Rightarrow$ unit vector $\vec{u}$ in direction of $\vec{v}$ is $\frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{\sqrt{6^2+15^2}}\vec{v}$

so $\vec{u} = \left\langle \frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}} \right\rangle$

f has continuous partial derivatives; hence $D_{\vec{u}}f(3,5) = \nabla f(3,5) \cdot \vec{u}$

$$= \left\langle -\frac{7}{3}, \frac{5}{3} \right\rangle \cdot \left\langle \frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}} \right\rangle$$

Hence

$$D_{\vec{u}} f(3,5) = \frac{-14}{3\sqrt{29}} + \frac{25}{3\sqrt{29}} = \frac{11}{3\sqrt{29}}$$

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3. (6+10=16 pts)

(a) Let $f(x,y) = 4x^3 - 6xy + y^2 + 2y$. Find and classify the critical points of $f(x,y)$.

To find critical points, set $\nabla f = \vec{0} = \langle 0,0\rangle = \langle f_x(x,y), f_y(x,y)\rangle$

$$\left.\begin{array}{l} f_x(x,y) = 12x^2 - 6y = 0 \\ f_y(x,y) = -6x + 2y + 2 = 0 \end{array}\right] \Rightarrow y = 2x^2 \ \& \ 4x^2 - 6x + 2 = 0 = (2x-2)(2x-1)$$

$$\Rightarrow \left[\begin{array}{l} x_1 = 1 \Rightarrow y_1 = 2 \\ x_2 = \frac{1}{2} \Rightarrow y_2 = \frac{1}{2} \end{array}\right\}$$ critical pts are $(1,2)$ and $(\frac{1}{2}, \frac{1}{2})$

To classify critical pts: $f_{xx}(x,y) = 24x$, $f_{yy}(x,y) = 2$, $f_{xy}(x,y) = -6$

Thus $\Delta(x,y) = f_{xx}(x,y)\cdot f_{yy}(x,y) - f_{xy}^2(x,y) = 48x - 36$

$\Delta(1,2) = 48 - 36 > 0$ & $f_{yy}(1,2) = 2 > 0 \Longrightarrow$ By 2^{nd} Derivative Test $(1,2)$ is a local minimum point of f

$\Delta(\frac{1}{2}, \frac{1}{2}) = 24 - 36 < 0 \Rightarrow$ By 2^{nd} D.T., $(\frac{1}{2}, \frac{1}{2})$ is a saddle point of f.

(b) Given that the extreme values exist, find the absolute maximum and minimum values of the function $f(x,y,z) = xyz$ subject to $xy + 2yz + 3xz = 18$ and $x \geq 0$, $y \geq 0$, $z \geq 0$.

Absolute Max/min points of f must be either on boundary points, or at critical pts or at singular pts.

- singular pt's: since f_1, f_2, f_3 exist everywhere, f has no singular pts.
- Boundary points: Let S be the surface $S: xy + 2yz + 3xz = 18$. Then

$$\left.\begin{array}{l} C_1 = S \cap (xy\text{-plane}) : z = 0,\ xy = 18 \\ C_2 : S \cap (yz\text{-plane}) : x = 0,\ yz = 9 \\ C_3 : S \cap (xz\text{-plane}) : y = 0,\ xz = 6 \end{array}\right\}$$ on each of C_1, C_2, C_3 we have $f(x,y,z) = xyz = 0 \quad \forall (x,y,z) \in C_i$, $i = 1,2,3$.

- To find critical pts of f on S, apply Lagrange Multipliers Mtd;

Find Max/min of $f(x,y,z) = xyz$

subject to $g(x,y,z) = xy + 2yz + 3xz - 18 = 0$

set $\nabla f = \lambda \nabla g$ and get

$$\left[\begin{array}{ll} yz = \lambda(y + 3z) & \text{--}(1) \\ xz = \lambda(x + 2z) & \text{--}(2) \\ xy = \lambda(2y + 3x) & \text{--}(3) \\ xy + 2yz + 3xz = 18 & \text{--}(4) \end{array}\right]$$ solve this system of 4 linear equations in 4 variables.

3(Eqn 2) − (Eqn 3): $3xz - xy = 3\lambda(x + 2z) - \lambda(2y + 3x)$

$x(3z - y) = 2\lambda(3z - y) \Rightarrow (3z - y)(x - 2\lambda) = 0$

$\Rightarrow \underline{y = 3z}$ or $\underline{x = 2\lambda}$

Case I: $(x = 2\lambda) \Rightarrow$ Eq 2: $2\lambda z = \lambda(2\lambda + 2z) \Rightarrow 2\lambda^2 = 0 \Rightarrow \lambda = 0$ contradiction.

Case II: $(y = 3z) \Rightarrow$ Eqn 1: $3z^2 = \lambda(6z) \Rightarrow 3z(z - 2\lambda) = 0 \Rightarrow \underline{z = 0}$ or $\underline{z = 2\lambda}$

- $z = 0 \Rightarrow y = 0 \Rightarrow x = 0 \Rightarrow$ contradicts Eqn (4).
- $z = 2\lambda \Rightarrow$ (Eqn 2): $x 2\lambda = \lambda(x + 4\lambda) \Rightarrow 2x = x + 4\lambda \Rightarrow x = 4\lambda = 2z \Rightarrow \boxed{x = 2z}$

$y = 3z$, $x = 2z \Rightarrow$ Eqn (4): $xy + 2yz + 3xz = 18 \Rightarrow 6z^2 + 2\cdot 3z^2 + 3\cdot 2z^2 = 18 \Rightarrow z^2 = 1 \Rightarrow z = \mp 1$

$z = -1 \Rightarrow (x,y,z) = (-2,-3,-1)$ Not in the domain.

$z = +1 \Rightarrow (x,y,z) = (2,3,1)$ is the only critical point in the domain. $(x \geq 0, y \geq 0, z \geq 0)$

candidates for Abs. Max./min:

Boundary pts: $f(x,y,z) = 0 \rightarrow$ Abs min, Critical pts $f(2,3,1) = 6$ Abs. MAX

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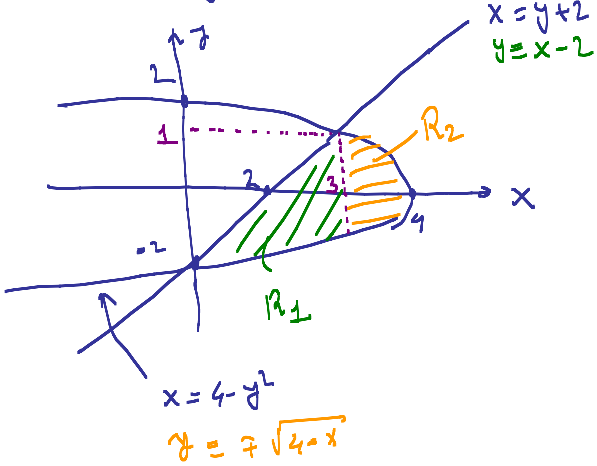
4.(14 pts) Sketch the region of integration and reverse the order of integration for the iterated integral $\displaystyle\int_{-2}^{1}\int_{y+2}^{4-y^2} \arctan\left(x^2\right)\,dx\,dy$ (Do **NOT** evaluate the integral.)

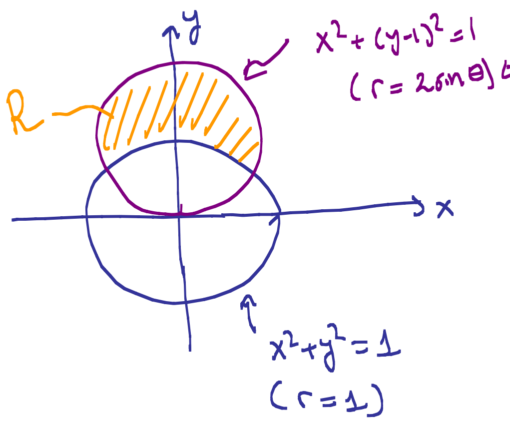
$$I = \int_{-2}^{1}\int_{y+2}^{4-y^2} \arctan(x^2)\,dx\,dy = \iint_R \arctan(x^2)\,dA \quad \text{where} \quad R = R_1 \sqcup R_2$$



$$= \iint_{R_1} \arctan(x^2)\,dA + \iint_{R_2} \arctan(x^2)\,dA$$

$$= \int_0^3 \int_{-\sqrt{4-x}}^{x-2} \arctan(x^2)\,dy\,dx + \int_3^4 \int_{-\sqrt{4-x}}^{+\sqrt{4-x}} \arctan(x^2)\,dy\,dx$$

5.(14 pts) Using a double integral, calculate the area of the region which lies inside the circle $x^2 + (y-1)^2 = 1$ and outside the circle $x^2 + y^2 = 1$.



$$\left[x^2+(y-1)^2=1 \Rightarrow x^2+y^2=2y \Rightarrow r^2 = 2r\sin\theta \Rightarrow r = 2\sin\theta\right]$$

Intersection pts: $\left.\begin{array}{l} r = 2\sin\theta \\ r = 1 \end{array}\right\}$ $1 = 2\sin\theta \Rightarrow \sin\theta = 1/2 \Rightarrow \theta = \frac{\pi}{6}$ or $\theta = \frac{5\pi}{6}$

$R:\ \frac{\pi}{6} \le \theta \le \frac{5\pi}{6}$
$\quad 1 \le r \le 2\sin\theta$

$$\text{Area}(R) = \iint_R 1.\,dy\,dx = \int_{\pi/6}^{5\pi/6}\int_1^{2\sin\theta} 1.\,r.\,dr\,d\theta = \int_{\pi/6}^{5\pi/6} \frac{r^2}{2}\Big|_1^{2\sin\theta}\,d\theta$$

$$= \int_{\pi/6}^{5\pi/6}\left(2\sin^2\theta - \frac{1}{2}\right)d\theta = \frac{1}{2}\int_{\pi/6}^{5\pi/6}(1-2\cos 2\theta)\,d\theta = \frac{1}{2}\Big[\theta - \sin 2\theta\Big]_{\pi/6}^{5\pi/6}$$

$$= \frac{1}{2}\left[\left(\frac{5\pi}{6} - \frac{\pi}{6}\right) + \left(\sin\frac{\pi}{3} - \sin\frac{5\pi}{3}\right)\right] = \frac{\pi}{3} + \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} = \frac{\pi}{3} + \frac{\sqrt{3}}{2}$$

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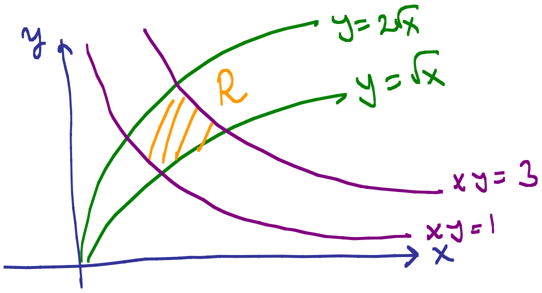
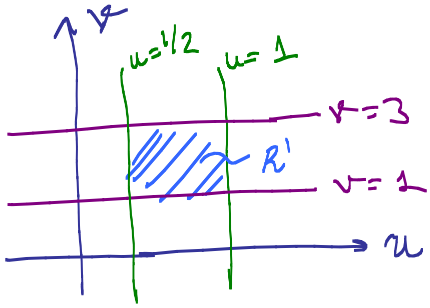
6.(15 pts) Evaluate the double integral $\displaystyle\iint_R \frac{\sqrt{x}}{y}\, e^{-\frac{\sqrt{x}}{y}}\, dA$ where R is the region on the $xy-$plane bounded by the curves $y = 2\sqrt{x},\ y = \sqrt{x},\ xy = 1,\ xy = 3.$



Let $u = \dfrac{\sqrt{x}}{y}$, $v = xy$



$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}} = \frac{1}{\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}} = \frac{1}{\begin{vmatrix} \frac{1}{2\sqrt{x}\,y} & \frac{-\sqrt{x}}{y^2} \\ y & x \end{vmatrix}} = \frac{1}{\frac{3}{2}\frac{\sqrt{x}}{y}} = \frac{2}{3u}$$

Thus

$$\iint_R f(x,y)\,dy\,dx = \iint_{R'} f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dv\,du$$

$$= \int_{1/2}^{1}\int_1^3 u e^{-u}\,\frac{2}{3u}\,dv\,du = \int_{1/2}^{1} \frac{2}{3} e^{-u} v \Big|_{v=1}^{v=3} du$$

$$= \int_{1/2}^{1} \frac{4}{3} e^{-u} du = \frac{4}{3}\left(-e^{-u}\right)\Big|_{1/2}^{1} = \frac{4}{3}\left(e^{-1/2} - e^{-1}\right)$$

7.(14 pts) Evaluate the triple integral $\displaystyle\iiint_S \frac{1}{\sqrt{x^2+y^2+z^2}}\,dV$ where

$$S = \{(x,y,z) \mid \sqrt{x^2+y^2} \le z \le \sqrt{3x^2+3y^2}, \quad 9 \le x^2+y^2+z^2 \le 16\}$$

that is, S is the region between the cones $z = \sqrt{x^2+y^2}$, $z = \sqrt{3x^2+3y^2}$ and also between the spheres $x^2+y^2+z^2 = 9$, $x^2+y^2+z^2 = 16$.

Using spherical coordinates ρ, ϕ, θ : $x^2+y^2+z^2 = 9 \iff \rho = 3$

$x^2+y^2+z^2 = 16 \iff \rho = 4$

$z = \sqrt{x^2+y^2} \iff z = \sqrt{r^2} \iff z = r \iff \rho\cos\phi = \rho\sin\phi$

$\cot\phi = 1 \iff \boxed{\phi = \pi/4}$

$z = \sqrt{3x^2+3y^2} \iff z = \sqrt{3r^2} \iff \cot\phi = \sqrt{3} \iff \boxed{\phi = \pi/6}$

equations of cones

$$S: \begin{bmatrix} 0 \le \theta \le 2\pi \\ \frac{\pi}{6} \le \phi \le \frac{\pi}{4} \\ 3 \le \rho \le 4 \end{bmatrix}$$

Hence

$$\iiint_S \frac{1}{\sqrt{x^2+y^2+z^2}}\,dV = \int_0^{2\pi}\int_{\pi/6}^{\pi/4}\int_3^4 \frac{1}{\sqrt{\rho^2}}\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= 2\pi\int_{\pi/6}^{\pi/4}\left(\frac{\rho^2}{2}\sin\phi\right)\Big|_{\rho=3}^{\rho=4} = 2\pi\int_{\pi/6}^{\pi/4}\frac{7}{2}\sin\phi\,d\phi$$

$$= 2\pi\left(-\frac{7}{2}\cos\phi\right)\Big|_{\phi=\pi/6}^{\phi=\pi/4} = 2\pi\,\frac{7}{2}\left(\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\right) = 7\pi\left(\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\right)$$

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M E T U
Department of Mathematics

	Calculus of Functions of Several Variables Second Final Exam	

Code : MATH 120 Acad. Year : 2015-2016 Semester : Spring Coord. : Muhiddin Uğuz	Last Name : Name : Department : Signature :	Student No. : Section :	
Date : 02.06.2016 Time : 9.30 Duration : 150 minutes	**8** Questions on **6** Pages Total **100** Points		

1	2	3	4	5	6	7	8	SHOW YOUR WORK	

1. (5+5=10 pts) Determine whether the followings are convergent or divergent.

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{\cos(n\pi)}{\sqrt{n}}$ $= \displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty}(-1)^n a_n$ is an alternating series since $\boxed{a_n=\frac{1}{\sqrt{n}}>0}$ $\forall n\geq 1$

$\cos(n\pi)=(-1)^n \ \forall n\in\mathbb{Z}$

• $\sqrt{n+1}>\sqrt{n} \Rightarrow \frac{1}{\sqrt{n+1}}<\frac{1}{\sqrt{n}}$

Hence $a_{n+1}<a_n \ \forall n\geq 1 \Rightarrow$ $\boxed{a_n \text{ is decreasing}}$

• $\boxed{\lim_{n\to\infty}a_n=\lim_{n\to\infty}\frac{1}{\sqrt{n}}=0}$

Thus by Alternating Series Test, given series is convergent.

(b) $a_n=\dfrac{(2n)!}{(n!)^2 4^n}$ $>0 \ \forall n\geq 1$

$$\frac{a_{n+1}}{a_n}=\frac{(2n+2)(2n+1)\cancel{(2n)!}}{(n+1)^2\cancel{(n!)^2}\cdot 4\cdot\cancel{4^n}}\cdot\frac{\cancel{(n!)^2}\cdot\cancel{4^n}}{\cancel{(2n)!}}=\frac{(2n+1)}{2(n+1)}=\frac{2n+1}{2n+2}<1 \ \forall n\geq 1 \Rightarrow a_n \text{ is decreasing}$$

a_n is bounded from below (by 0) and decreasing hence $\{a_n\}$ is a convergent sequence

2. (5+5=10 pts) Let f be a function which has a power series representation centred at 2 such that $f(2)=3$, $f'(2)=7$, $f''(2)=0$, $f'''(2)=120$. Let g be a function which is continuous but not differentiable at 2 with $g(2)=119$.

(a) Find the first three non zero terms of the Taylor series of $f(x)$ centred at 2.

Taylor series of f centered at 2 is $\displaystyle\sum_{n=0}^{\infty}\frac{f^{(n)}(2)}{n!}(x-2)^n=f(2)+f'(2)(x-2)+\frac{f^{(2)}(2)}{2!}(x-2)^2+\dots$

First 3 nonzero terms of this Power Series is: $3+7(x-2)+\dfrac{120}{3!}(x-2)^3$

(b) Evaluate $\displaystyle\lim_{x\to 2}\frac{f(x)-3-7(x-2)}{(x-2)^2\,g(x)}$ (Note that you cannot use L'Hospital's rule since $g(x)$ is not differentiable at 2).

$f(x)=3+7(x-2)+20(x-2)^3+$ higher order terms.

$$\lim_{x\to 2}\frac{f(x)-3-7(x-2)}{(x-2)^2\cdot g(x)}=\lim_{x\to 2}\frac{20(x-2)^3+\sum_{n=4}^{\infty}\frac{f^{(n)}(2)(x-2)^n}{n!}}{(x-2)^2\cdot g(x)}=\lim_{x\to 2}\frac{20(x-2)+\sum_{n=4}^{\infty}\frac{f^{(n)}(2)(x-2)^{n-2}}{n!}}{g(x)}$$

$$=\lim_{x\to 2}\frac{20\cdot 0+0}{119}=0$$

3. (5+5=10 pts)

(a) Find the angle of intersection of the surfaces $z = 2x^2 + y^2$ and $8x + 6y - z = 5$ at the point $(4, 1, 33)$.(The angle between two surfaces at a point is the angle between their tangent planes at that point).

The surface $z = 2x^2 + y^2$ is level surface of $F(x,y,z) = 2x^2 + y^2 - z$ $(F(x,y,z) = 0)$. Here $F(x,y,z)$ has continuous partial derivatives and hence differentiable.

Thus $\nabla F(4,1,33)$ is normal to the surface $F=0$ at $(4,1,33)$.

$\nabla F(x,y,z) = \langle 4x, 2y, -1\rangle \Rightarrow \vec{n}_1 = \langle 16, 2, -1\rangle$

Similarly $8x+6y-z=5$ is level surface of $G(x,y,z) = 8x+6y-z$ and $\nabla G = \langle 8,6,-1\rangle$

Thus $\vec{n}_2 = \langle 8, 6, -1\rangle$

$\vec{n}_1 \cdot \vec{n}_2 = \|\vec{n}_1\| \|\vec{n}_2\| \cos\theta \Rightarrow 16\cdot 8 + 2\cdot 6 + (-1)(-1) = 141 = \sqrt{16^2+2^2+(-1)^2}\sqrt{8^2+6^2+(-1)^2}\cos\theta$

$$\theta = \arccos\left(\frac{141}{\sqrt{261}\sqrt{101}}\right)$$

(b) Find a parametrization of the curve of intersection of the surfaces $z = 2x^2 + y^2$ and $8x + 6y - z = 5$.

Let C be the curve of intersection. $(x,y,z) \in C \iff z = 2x^2+y^2$ and $z = 8x+6y-5$

$2x^2+y^2 = 8x+6y-5 \Rightarrow 2(x^2-4x) + y^2 - 6y = -5 \Rightarrow 2((x-2)^2-4) + (y-3)^2 - 9 = -5$

$\Rightarrow 2(x-2)^2 + (y-3)^2 = 12 \Rightarrow \frac{(x-2)^2}{6} + \frac{(y-3)^2}{12} = 1$ is an ellipse in the xy-plane. It is projection of C to xy-plane.

Parametrize ellipse

Let $x = 2+\sqrt{6}\cos\theta$, $y = 3+2\sqrt{3}\sin\theta$, $\theta \in [0,2\pi]$

$z = 8x+6y-5 = 8(2+\sqrt{6}\cos\theta) + 6(3+2\sqrt{3}\sin\theta) - 5$

Thus parametrization of C is

$C \leftrightarrow \vec{r}(t) = (2+\sqrt{6}\cos\theta, 3+2\sqrt{3}\sin\theta, 29+8\sqrt{6}\cos\theta+12\sqrt{3}\sin\theta)$ $\theta \in [0,2\pi]$

4. (4+3= 10 pts)

A differentiable function $z(x, y)$ is defined implicitly by $2z + x + y + e^z = 3$.

(a) Find the direction in which $z(x, y)$ decreases most rapidly at the point $(x, y) = (1, 1)$ when $z = 0$.

$\frac{\partial}{\partial x}(2z+x+y+e^z) = 2z_x + 1 + z_x e^z = 0 \Rightarrow z_x = -1/(2+e^z)$

$\frac{\partial}{\partial y}(2z+x+y+e^z) = 2z_y + 1 + z_y e^z = 0 \Rightarrow z_y = -1/(2+e^z)$

$(x,y) = (1,1) \Rightarrow 2z+1+1+e^z = 3 \Rightarrow 2z+e^z = 1 \Rightarrow z = 0$

at $(x,y,z) = (1,1,0)$, $z_x = -1/(2+e^0) = -1/3$, $z_y = -1/(2+e^0) = -1/3$

Thus $\nabla z(1,1) = \langle z_x(1,1), z_y(1,1)\rangle = -\frac{1}{3}\langle 1,1\rangle$

z decreases the most rapidly in direction of the vector $-\nabla z(1,1) = \frac{1}{3}\langle 1,1\rangle$

Thus The direction in which z decreases the most rapidly is $\frac{\frac{1}{3}\langle 1,1\rangle}{\|\frac{1}{3}\langle 1,1\rangle\|} = \frac{1}{\sqrt{2}}\langle 1,1\rangle$

(b) Are there points where the largest rate of change is greater than $\sqrt{2}$? Explain.

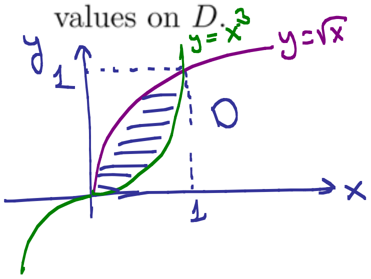
$\nabla z(x,y) = \langle z_x(x,y), z_y(x,y)\rangle = \left\langle \frac{-1}{2+e^z}, \frac{-1}{2+e^z}\right\rangle$

$\|\nabla z(x,y)\|$ = max rate of change of $z(x,y)$ at (x,y)

$\left\| \frac{-1}{2+e^z}\langle 1,1\rangle\right\| = \frac{\sqrt{2}}{2+e^z} \lneq \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}} \lneq \sqrt{2}$. Therefore, there are NO such points

5.(3+12=15 pts) Let $f(x,y) = 8x^3y - 3x - y^2$ and let D be the region in the first quadrant given by $D = \{(x,y) \mid x^3 \le y \le \sqrt{x}, x \ge 0\}$.

(a) Sketch the region D and explain why $f(x,y)$ has absolute maximum and minimum values on D.



D is closed and bounded, f is continuous on D. Thus by Extreme Value Theorem, f has an absolute MAX and absolute min. value on D.

(b) Find the absolute extreme values of $f(x,y)$ on D.

Extreme values (they exist by part (a)) can be one of
- Singular pts. of f inside D (but f_x, f_y are defined everywhere in D so no sing. pts.)
- critical pts of f inside D
- Boundary pts. of D.

Critical pts: $\nabla f = \vec{0} = \langle 0,0\rangle = \langle f_x(x,y) = f_y(x,y)\rangle$

$f_x = 24x^2y - 3 = 0$, $f_y = 8x^3 - 2y = 0$ $\quad x^2y = 1/8 \Rightarrow y = 1/8x^2$, $4x^3 = y = \frac{1}{8x^2} \Rightarrow x^5 = \frac{1}{32} \Rightarrow x = \frac{1}{2} \Rightarrow y = \frac{1}{8(\frac{1}{2})^2} = \frac{1}{2}$

Thus the only critical pt is $(\frac{1}{2}, \frac{1}{2})$ and it is inside D.

$\boxed{f(\tfrac{1}{2}, \tfrac{1}{2}) = -5/4}$

Boundary pts

$\partial D = C_1 \sqcup C_2$ where $C_1: (x, x^3) : x \in [0,1]$
$C_2: (x, \sqrt{x}) : x \in [0,1]$

on C_1: $f(x,y) = f(x,x^3) = 8x^6 - 3x - x^6 = 7x^6 - 3x = g(x)$; $x \in [0,1]$

- critical pts of g: $g'(x) = 42x^5 - 3 = 0 \Rightarrow x = (\frac{1}{14})^{1/5}$

$g((\frac{1}{14})^{1/5}) = \boxed{f((\tfrac{1}{14})^{1/5}, (\tfrac{1}{14})^{3/5}) = \tfrac{-5}{2}(\tfrac{1}{14})^{1/5}}$

- end pts $g(0) = \boxed{f(0,0) = 0}$, $g(1) = \boxed{f(1,1) = 4}$

on C_2

$f(x,y) = f(x, \sqrt{x}) = 8x^3\sqrt{x} - 3x - x = 8x^{7/2} - 4x = h(x)$; $x \in [0,1]$

- critical pts of h: $h'(x) = 8 \cdot \frac{7}{2} x^{5/2} - 4 = 0 \Rightarrow x = (\frac{1}{7})^{2/5}$ is the only critic. pt. inside $(0,1)$

$h((\frac{1}{7})^{2/5}) = \boxed{f((\tfrac{1}{7})^{2/5}, (\tfrac{1}{7})^{1/5}) = \tfrac{-20}{7}(\tfrac{1}{7})^{2/5}}$

- end pts $\boxed{h(0) = f(0,0) = 0}$, $\boxed{h(1) = f(1,1) = 4}$

The only candidates for the abs. MAX and abs. min of f on D are the ones in the boxes above. Thus

Abs. MAX $= 4 = f(1,1)$ and

Abs. min $= -\frac{5}{2}(\frac{1}{14})^{1/5} = f((\frac{1}{14})^{1/5}, (\frac{1}{14})^{3/5})$

6.(6+5+4=15 pts) Let $\vec{F}$ be the vector field given by

$$\vec{F}(x,y,z) = \langle y^2\cos(xy^2) + yz^2\ ,\ 2xy\cos(xy^2) + xz^2 + ze^{yz}\ ,\ 2xyz + ye^{yz} + 2z\rangle$$

$$\left(= (y^2\cos(xy^2) + yz^2)\,\vec{i}\ +\ (2xy\cos(xy^2) + xz^2 + ze^{yz})\,\vec{j}\ +\ (2xyz + ye^{yz} + 2z)\,\vec{k}\right)$$

(a) Show that $\vec{F}$ is conservative on $\mathbb{R}^3$

$\vec{F}$ is conservative on $\mathbb{R}^3$ $\iff$ $\vec{F} = \nabla f$ on $\mathbb{R}^3$ (f: potential function)

$$\vec{F} = \nabla f \iff \begin{cases} f_x = y^2\cos(xy^2) + yz^2 \\ f_y = 2xy\cos(xy^2) + xz^2 + ze^{yz} \\ f_z = 2xyz + ye^{yz} + 2z \end{cases}$$

$f_x = y^2\cos(xy^2) + yz^2 \Rightarrow f(x,y,z) = \int (y^2\cos(xy^2) + yz^2)\,dx = \sin(xy^2) + xyz^2 + C(y,z)$

$\Rightarrow f_y(x,y,z) = 2xy\cos(xy^2) + xz^2 + C_y(y,z)$
$= 2xy\cos(xy^2) + xz^2 + z\,e^{yz}$

$\Rightarrow C_y(y,z) = ze^{yz} \Rightarrow C(y,z) = \int ze^{yz}\,dy = e^{yz} + D(z)$

$\Rightarrow f(x,y,z) = \sin(xy^2) + xyz^2 + e^{yz} + D(z)$

$\Rightarrow f_z = 2xyz + ye^{yz} + D'(z) = 2xyz + ye^{yz} + ye^{yz} + 2z \Rightarrow D'(z) = 2z$

$\Rightarrow D(z) = z^2 + E$ constant.

Thus

$f(x,y,z) = \sin(xy^2) + xyz^2 + e^{yz} + z^2 + E$ $(\Rightarrow \nabla f = \vec{F})$

is a potential function for vector field $\vec{F}$, for any constant E.

$\therefore$ $\vec{F}$ is a conservative vector field on $\mathbb{R}^3$

(b) Let C be the curve of intersection of the cone $z^2 = 4x^2 + 9y^2$ and the plane $z = 1 + x + 2y$. D is defined as the part of the curve C that lies in the first octant $(x \geq 0, y \geq 0, z \geq 0)$ from the point $(1,0,2)$ to $(0,1,3)$. Calculate $\displaystyle\int_D \vec{F}\cdot d\vec{r}$.

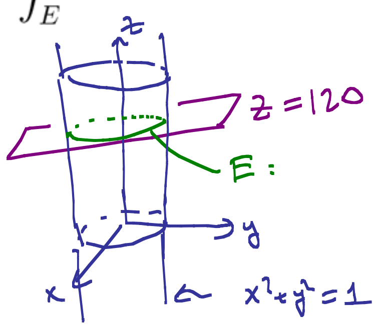
Since $\vec{F}$ is conservative, and $\vec{F} = \nabla f$ from part (a), we have

$$\int_D \vec{F}\cdot d\vec{r} = \int_D \nabla f \cdot d\vec{r} = f(B) - f(A) = f(0,1,3) - f(1,0,2)$$

A: initial pt of D : (1,0,2)
B: final " " " : (0,1,3)

$$= (\sin(0) + 0 + e^3 + 9 + E) - (\sin 0 + 0 + e^0 + 4 + E)$$
$$= e^3 + 4$$

(c) Let E be the curve given by the intersection of $x^2 + y^2 = 1$ and $z = 120$. Calculate $\displaystyle\int_E \vec{F}\cdot d\vec{r}$.



E is a circle, and hence a closed curve
(initial pt of E and final pt of E are the same)

Since E is closed and $\vec{F}$ is conservative, we have

$$\int_E \vec{F}\cdot d\vec{r} = 0$$

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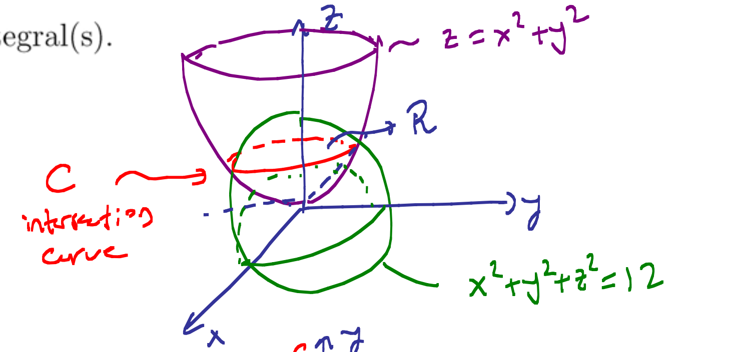
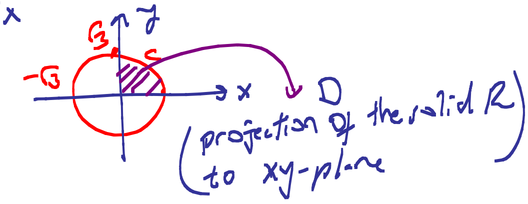
7.(5+5+5=15 pts) Let R be the solid in the first octant ($x \geq 0, y \geq 0, z \geq 0$) which lies inside $x^2 + y^2 + z^2 = 12$and above $z = x^2 + y^2$. Express the triple integral $\iiint_R f(x, y, z)\, dV$ as a (sum of) iterated integral(s).





(a) in cartesian coordinates

$x^2+y^2+z^2=12$ and $z=x^2+y^2$

$\Rightarrow z+z^2=12 \Rightarrow (z+4)(z-3)=0$

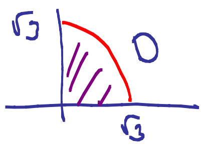
$\Rightarrow z=3$ on C

$$R: \begin{cases} 0 \le x \le \sqrt{3} \\ 0 \le y \le \sqrt{3-x^2} \\ x^2+y^2 \le z \le \sqrt{12-x^2-y^2} \end{cases}$$

Thus

$$\iiint_R f(x,y,z)\,dV = \int_0^{\sqrt{3}} \int_0^{\sqrt{3-x^2}} \int_{x^2+y^2}^{\sqrt{12-x^2-y^2}} f(x,y,z)\,dz\,dy\,dx$$

(b) in cylindrical coordinates



$$\left.\begin{array}{l} 0 \le \theta \le \pi/2 \\ 0 \le r \le \sqrt{3} \end{array}\right\} \Rightarrow R: \begin{array}{l} 0 \le \theta \le \pi/2 \\ 0 \le r \le \sqrt{3} \\ r^2 \le z \le \sqrt{12-r^2} \end{array}$$

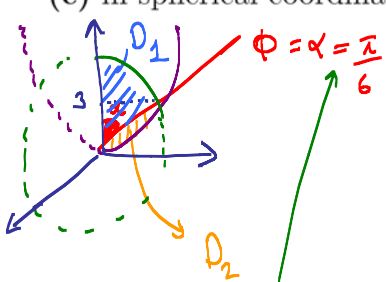
$z = x^2+y^2 \longleftrightarrow z=r^2$

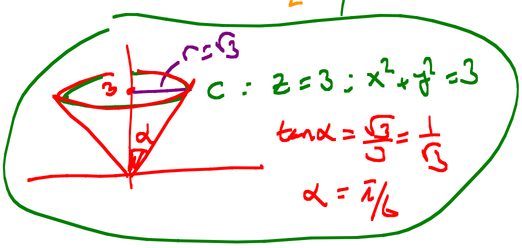
$x^2+y^2+z^2=12 \longleftrightarrow r^2+z^2=12 \Rightarrow z = \mp\sqrt{12-r^2}$ & $z \geq 0 \Rightarrow z=\sqrt{12-r^2}$

Thus

$$\iiint_R f(x,y,z)\,dV = \int_0^{\pi/2} \int_0^{\sqrt{3}} \int_{r^2}^{\sqrt{12-r^2}} f(r\cos\theta, r\sin\theta, z)\, r\,dz\,dr\,d\theta$$

(c) in spherical coordinates





$x \geq 0,\ y \geq 0 \Rightarrow 0 \le \theta \le \pi/2$

For D_1 : $0 \le \phi \le \pi/6$

For D_2 : $\frac{\pi}{6} \le \phi \le \pi/2$

$x^2+y^2+z^2=12 \Rightarrow \rho^2=12 \Rightarrow \rho = 2\sqrt{3}$

$z = x^2+y^2 \Rightarrow \rho\cos\phi = r^2 = \rho^2\sin^2\phi \Rightarrow \rho = \frac{\cos\phi}{\sin^2\phi}$

Thus

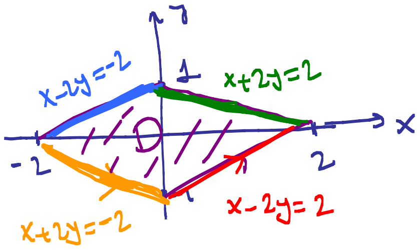
$$\iiint_R f(x,y,z)\,dV = \int_0^{\pi/2}\int_0^{\pi/6}\int_0^{2\sqrt{3}} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$+ \int_0^{\pi/2}\int_{\pi/6}^{\pi/2}\int_0^{\frac{\cos\phi}{\sin^2\phi}} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

8.(15 pts) Compute the line integral

$$\oint_C \left(x^3 \sin\sqrt{x^2+4} - xe^{x+2y}\right)\, dx + \left(\cos\left(y^3+y\right) - 4ye^{x+2y}\right)\, dy$$

on the curve C which is the boundary of the parallelogram with vertices $(2,0), (0,1), (-2,0), (0,-1)$ with counter-clockwise orientation.



$\vec{F}(x,y) = (P(x,y), Q(x,y))$ where

$P(x,y) = x^3 \sin\sqrt{x^2+4} - x e^{x+2y}$

$Q(x,y) = \cos(y^3+y) - 4y e^{x+2y}$

P & Q have continuous partial derivatives on D, $C = \partial D$ is the counterclockwise oriented boundary curve of D. Moreover C is closed and piecewise smooth curve.

Hence, by Green's Thm;

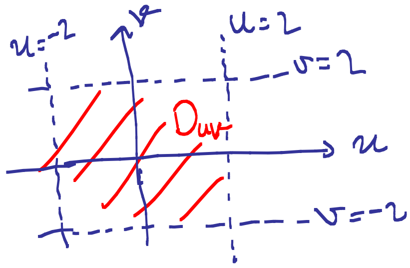
$$\oint_{C=\partial D} P(x,y)\,dx + Q(x,y)\,dy = \iint_D \left[Q_x(x,y) - P_y(x,y)\right] dA$$

$$= \iint_D \left[-4y e^{x+2y} - (-x e^{x+2y} \cdot 2)\right] dA$$

$$= \iint_D 2(x-2y) e^{x+2y}\, dx\, dy. \quad \text{Let } \begin{array}{l} u = x-2y \\ v = x+2y \end{array}$$

then

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}} = \frac{1}{\begin{vmatrix} 1 & -2 \\ 1 & +2 \end{vmatrix}} = \frac{1}{4}$$



Hence

$$\iint_{D_{xy}} 2(x-2y) e^{x+2y}\, dx\, dy = \iint_{D_{uv}} 2u e^{v} \cdot \frac{1}{4}\, du\, dv$$

$$= \int_{-2}^{2} \int_{-2}^{2} 2u \cdot e^{v} \frac{1}{4}\, du\, dv = \int_{-2}^{2} \frac{u^2}{4} e^{v} \Big|_{u=-2}^{u=+2} dv = \int_{-2}^{2} 0\, dv = 0$$